

THE FAR INTERNAL GRAVITY WAVE FIELD IN STRATIFIED MEDIA OF NON-UNIFORM DEPTH

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ABSTRACT

Uniform asymptotic forms of the far field of internal gravity waves generated by a source moving above a smoothly varying bottom are constructed. The solution is proposed in terms of wave modes, propagating independently in the adiabatic approximation, and described as a non-integer power series of a small parameter characterizing the stratified medium. A specific form of the wave packets, which can be parameterized in terms of model functions, depends on a local behavior of the dispersion curves of individual wave modes. A modified space-time ray method is proposed, which belongs to the class of geometrical optics methods. The key point of the proposed technique is the possibility to derive the asymptotic representation of the solution in terms of a non-integer power series of a small parameter.

INTRODUCTION

As is well known, an essential influence of the propagation of internal gravity waves in stratified natural media (ocean, atmosphere) is caused by the horizontal inhomogeneity and nonstationarity of these media. To the most typical horizontal inhomogeneities of a real ocean one can refer the modification of the relief of the bottom, and inhomogeneity of the density field, and the variability of the mean flows. One can obtain an exact analytic solution of this problem (for instance, by using the method of separation of variables) only if the distribution of density and the shape of the bottom are described by rather simple model functions. If the shape of the bottom and the stratification are arbitrary, then one can construct only asymptotic representation of the solution in the near and far zones; however, to describe the field of internal waves between these zones, one needs an accurate numerical solution of the problem.

Using asymptotic methods, one can consider a wide class of interesting physical problems, including problems concerning the propagation of nonharmonic wave packets of internal gravity waves in diverse nonhomogeneous stratified media under the assumption that the modification of the parameters of a vertically stratified medium are slow in the horizontal direction. From the general point of view, problems of this kind can be studied in the framework of a combination of the adiabatic and semiclassical approximations or by using close approach, for example, ray expansions. In particular, the asymptotic solutions of diverse dynamical problems can be described by using the Maslov canonical operator, which determines the asymptotic behavior of the solution, including the case of neighborhoods of singular sets composed of focal points, caustics, etc. The specific form of the wave packet can be finally expressed by using some special functions, say, in terms of oscillating exponentials, Airy function, Fresnel integral, Pearcey-type integral, etc. The above approaches are quite general and, in principle, enable one to solve a broad spectrum of problems from the mathematical point of view; however, the problem of their practical applications and, in particular, of the visualization of the corresponding asymptotic formulas based on the Maslov canonical operator is still far from completion, and in some specific problems to find the asymptotic behavior whose computer realization using software of *Mathematica* type is rather simple. In this paper, using the approaches developed in [1-3], we construct and numerically realize asymptotic solutions of the problem, which is formulated as follows.

BASIC EQUATIONS

In this study we consider a non-viscous incompressible nonhomogeneous liquid. If it is unperturbed, we denote its density by $\rho(z)$ (the stratification is supposed to be stable, i.e. $\partial\rho/\partial z < 0$, the axis z is directed downward from the liquid surface). The system of the hydrodynamic equations

takes the following form [1]:
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}, \quad \text{div } \mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} = 0,$$

where $\mathbf{v} = (u, v, w)$ is the velocity vector, $\mathbf{g} = (0, 0, g)$ is the gravitation acceleration vector, p and ρ are the deviations of the pressure and the density from their equilibrium values. We consider a liquid layer with Brunt-Vaisaila frequency $N^2(z) = -g\partial \ln \rho / \partial z$. Let this layer be bounded by the surface $z=0$ and the bottom $z = \beta y$ and let the point mass source move at depth z_0 uniformly and rectilinearly with the velocity V in the negative direction of the abscissa axis along the line $y = y_0$. Then the velocity field in the Boussinesq approximation satisfies the following system of linearized equations:

$$\frac{\partial^2}{\partial t^2} \left(\Delta w + \frac{\partial^2}{\partial z^2} w \right) + N^2(z) \Delta w = \delta''_u(x+Vt) \delta(y-y_0) \delta'(z-z_0),$$

$$\Delta u + \frac{\partial^2 w}{\partial x \partial z} = \delta'(x+Vt) \delta(y-y_0) \delta(z-z_0), \quad \Delta v + \frac{\partial^2 w}{\partial y \partial z} = \delta(x+Vt) \delta'(y-y_0) \delta(z-z_0), \quad \text{where}$$

$$\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2, \quad w=0 \text{ at } z=0, \quad w = u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y}, \text{ at } z = H(x, y). \text{ We introduce the char-}$$

acteristic vertical scale L , and the ratio $\varepsilon = N^2 L / g \equiv \lambda^{-1} \ll 1$; in the real ocean one has $\varepsilon \sim 10^{-2}$.

When $N^2(z) = N^2 = \text{const}$ and the depth H is constant, we have $\varepsilon = N^2 H^2 / gH = \pi^2 c_{\text{max}}^2 / c^{-2}$, here

$c = \sqrt{gH}$ is the velocity of the long waves on the surface of liquid with depth H , and $c_{\text{max}} = NH / \pi$

is the maximal group velocity of the internal waves. In the real ocean one has $c \sim 100$ m/sec, and

$c_{\text{max}} \sim 1$ m/sec [1]. We seek the solutions of linearized system in the form of the sum of wave

modes [2,3]: $w = \sum_{n=1}^{\infty} w^n, v = \sum_{n=1}^{\infty} v^n$ (adiabatic approximation). Further we consider only the dominant

(first) mode omitting the upper index n . We seek the solution of the form

$$w = \int F(z, y, \omega) \exp i \varphi(\omega, y) d\omega, \quad v = \int \Psi(z, y, \omega) \exp \varphi(\omega, y) d\omega, \quad \varphi(\omega, y) = \lambda(\omega \xi - S(y, \omega)), \quad \text{where}$$

$$\xi = x + Vt, \Psi = i\Psi_1(z, y, \omega) + O(\lambda^{-1}), \quad F = F_0(z, y, \omega) + i\lambda^{-1} F_1(z, y, \omega) + O(\lambda^{-2}). \text{ We emphasize that}$$

functions w and v depend on ξ only via the phases of elementary solutions, superposition of

which gives the complete solutions F and Ψ . Therefore the problem is reduced to finding functions

$F(z, y, \omega)$ and $S(y, \omega)$. The function $F_0(z, y, \omega)$ are obtained from the Sturm-Liouville problem:

$$\frac{\partial^2 F_0}{\partial z^2} + \frac{\omega^2 + (\partial S / \partial y)^2}{\omega^2} (1 - \omega^2) F_0 = 0, \quad F_0 = 0 \text{ at } z = 0, H(y). \text{ The quantization condition yields the}$$

$$\text{eikonal equation, } \omega^2 + (\partial S / \partial y)^2 = k^2(\omega, y), \text{ the dispersion relation, } k^2(\omega, y) = \frac{\omega^2}{(1 - \omega^2)y^2} \text{ and the}$$

eigenfunction $F_0(z, y, \omega) = c(y, \omega) \sin \frac{zy_0}{y}$, where $c(y, \omega)$ is yet an arbitrary function which does

not depend on z and is obtained from the conservation law and the locality

MAIN RESULTS AND DISCUSSION

The wave field in the domain of the subcritical velocity ($y < y_0$) may be obtained by the stationary

$$\text{phase } (\lambda \gg 1) \text{ method: } w(\xi, y) = \frac{B(\omega, y, y_0, z, z_0)}{\sqrt{\lambda \frac{\partial \xi(y, \omega)}{\partial \omega}}} \cos \left[\varphi(\omega, y) + \frac{\pi}{4} \right], \text{ where } B(\omega, y, y_0, z, z_0) \text{ is}$$

some amplitude function, $\xi(y, \omega) = \frac{\partial S(y, \omega)}{\partial \omega}$ is the stationary point. The wave field in the domain

of the overcritical velocities ($y > y_0$) may be obtained by using the uniform approximation for the integrand function in the form

$$F(\omega, \xi, y) = \frac{A(\omega, y, z, z_0) \cos(\varphi(\omega, y)) \left(\frac{3}{2} \Delta(\omega, y) \right)^{1/6}}{\sqrt{1 - y^2 \alpha}} Ai \left(\left(\frac{3}{2} \Delta(\omega, y) \right)^{2/3} \right)$$

$$\Delta(\omega, y) = \frac{\omega}{\sqrt{\alpha}} \left(\ln \frac{1}{\sqrt{\alpha}} - \sqrt{1 - y^2 \alpha} + \ln(1 + \sqrt{1 - y^2 \alpha}) - \ln y \right)$$

where $A(\omega, y, z, z_0)$ is an amplitude function, $Ai(x)$ is the Airy function.

In Fig.1 we represent the rays' families for values of $y_0 = 0.3$. There are the ascending rays possessing a turning point and the descending ones without a turning point. The envelope curve for the rays (the caustic surface) is also represented in the Figure. In Fig. 2,3 we represent vertical component of internal gravity wave field velocity w obtained by integration of this uniform Airy asymptotic form.

The asymptotic representations constructed in this paper allow one to describe the far field of the internal gravity waves generated by a source moving over a slowly varying bottom. The obtained asymptotic expressions for the solution are uniform and reproduce fairly well the essential features of wave fields near caustic surfaces and wave fronts. In this paper the problem of reconstructing non-harmonic wave packets of internal gravity waves generated by a source moving in a horizontally stratified medium is considered. The solution is proposed in terms of modes, propagating independently in the adiabatic approximation, and described as a non-integer power series of a small parameter characterizing the stratified medium. In this study we analyze the evolution of non-harmonic wave packets of internal gravity waves generated by a moving source under the assumption that the parameters of a vertically stratified medium (e.g. an ocean) vary slowly in the horizontal direction, as compared to the characteristic length of the density. A specific form of the wave packets, which can be parameterized in terms of model functions, e.g. Airy functions, depends on local behavior of the dispersion curves of individual modes in the vicinity of the corresponding critical points.

In this paper a modified space-time ray method is proposed, which belongs to the class of geometrical optics methods [1-3]. The key point of the proposed technique is the possibility to derive the asymptotic representation of the solution in terms of a non-integer power series of the small parameter $\varepsilon = \lambda/L$, where λ is the characteristic wave length, and L is the characteristic scale of the horizontal heterogeneity. The explicit form of the asymptotic solution was determined based on the principles of locality and asymptotic behavior of the solution in the case of a stationary and horizontally homogeneous medium. The wave packet amplitudes are determined from the energy conservation laws along the characteristic curves. A typical assumption made in studies on the internal wave evolution in stratified media is that the wave packets are locally harmonic. A modification of the geometrical optics method, based on an expansion of the solution in model functions, allows one to describe the wave field structure both far from and at the vicinity of the wave front.

Using the asymptotic representation of the wave field at a large distance from a source moving in a layer of constant depth, we solve the problem of constructing the uniform asymptotics of the internal waves in a medium of varying depth. The solution is obtained by modifying the previously proposed "vertical modes-horizontal rays" method, which avoids the assumption that the medium parameters vary slowly in the vertical direction. The solution is parameterized through the Airy waves. This allows one to describe not only the evolution of the non-harmonic wave packets propagating over a slow-varying fluid bottom, but also specify the wave field structure associated with an individual mode both far from and close to the wave front of the mode. The Airy function argument is determined by solving the corresponding eikonal equations and finding vertical spectra of the internal gravity waves. The wave field amplitude is determined using the energy conservation law, or another adiabatic invariant, characterizing wave propagation along the characteristic curves.

Modeling typical shapes and stratification of the ocean shelf, we obtain analytic expressions describing the characteristic curves and examine characteristic properties of the wave field phase structure. As a result it is possible to observe some peculiarities in the wave field structure, depend-

ing on the shape of ocean bottom, water stratification and the trajectory of a moving source. In particular, we analyze a spatial blocking effect of the low-frequency components of the wave field, generated by a source moving alongshore with a supercritical velocity. Numerical analyses that are performed using typical ocean parameters reveal that actual dynamics of the internal gravity waves are strongly influenced by horizontal nonhomogeneity of the ocean bottom. In this paper we use an analytical approach, which avoids the numerical calculation widely used in analysis of internal gravity wave dynamics in stratified ocean.

In this paper the most difficult question is considered that can appear when we investigate the problems of wave theory with the help of geometrical optics methods and its modifications. And the main question consists in finding of asymptotic solution near special curve (or surface), which is called caustic. It is well known, that caustic is an envelope of a family of rays, and asymptotic solution is obtained along these rays. Asymptotic representation of the field describe qualitative change of the wave field, and that is description of the field, when we cross the area of “light”, where wave field exists, and come in the area of “shadow”, where we consider wave field to be rather small. Each point of the caustic corresponds to a specified ray, and that ray is tangent at this point.

It is a general rule that caustic of a family of rays single out an area in space, so that rays of that family cannot appear in the marked area. There is also another area, and each point of that area has two rays that pass through this point. One of those rays has already passed this point, and another is going to pass the point. Formal approximation of geometrical optics or WKB approximation cannot be applied near the caustic, that is because rays merge together in that area, after they were reflected by caustic. If we want to find wave field near the caustic, then it is necessary to use special approximation of the solution, and in the paper a modified ray method is proposed in order to build uniform asymptotic expansion of integral forms of the internal gravity wave field. After the rays are reflected by the caustic, there appears a phase shift. It is clear that the phase shift can only happen in the area where methods of geometrical optics, which were used in previous sections, can't be applied. If the rays touch the caustic several times, then additional phase shifts will be added. Phase shift, which was created by the caustic, is rather small in comparison with the change in phase along the ray, but this shift can considerably affect interference pattern of the wave field.

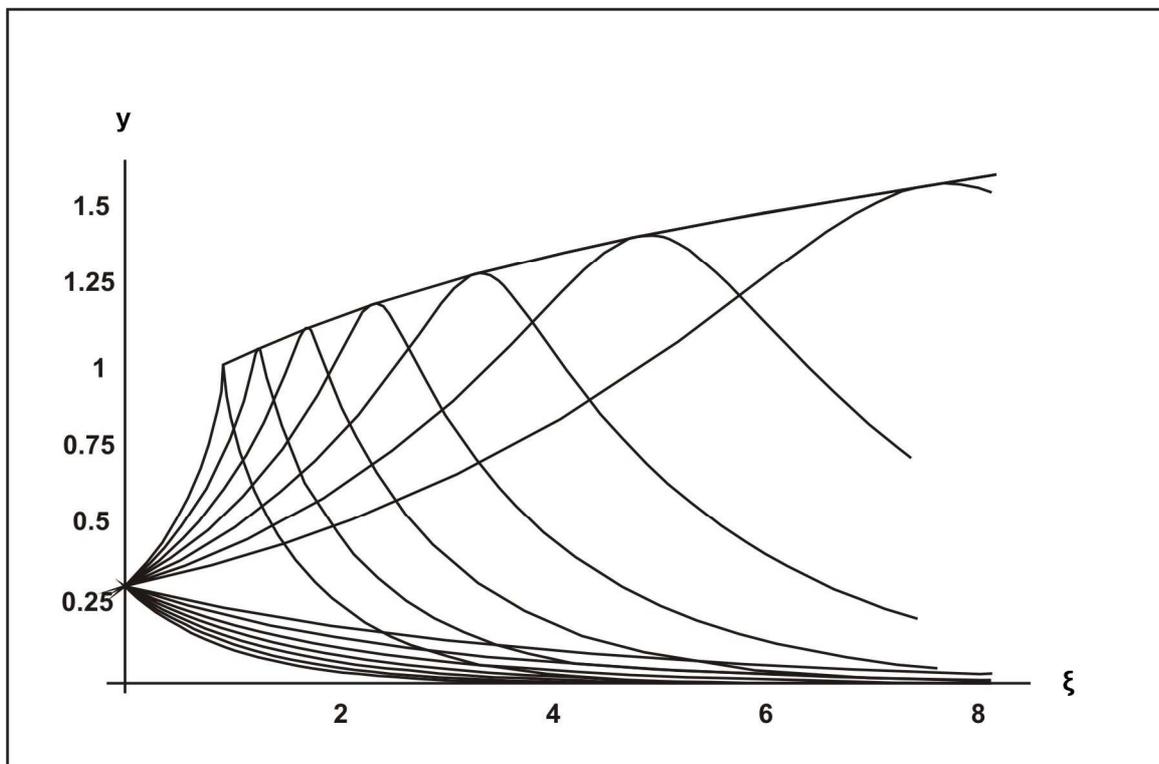


Fig. 1 Rays and wave fronts from a source moving in stratified media of variable depth. in non-dimensional coordinates ($\xi = \xi/H, y = y/H$) plane.

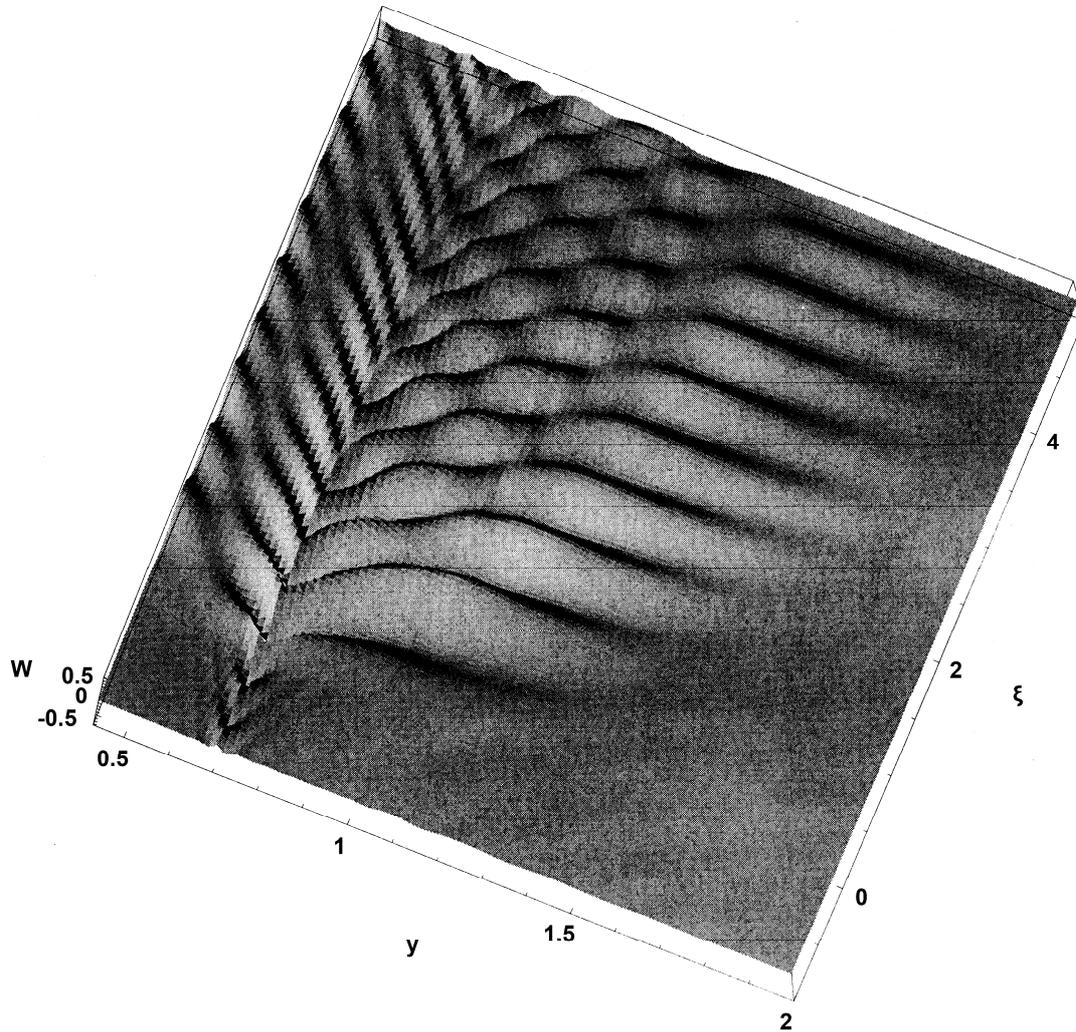
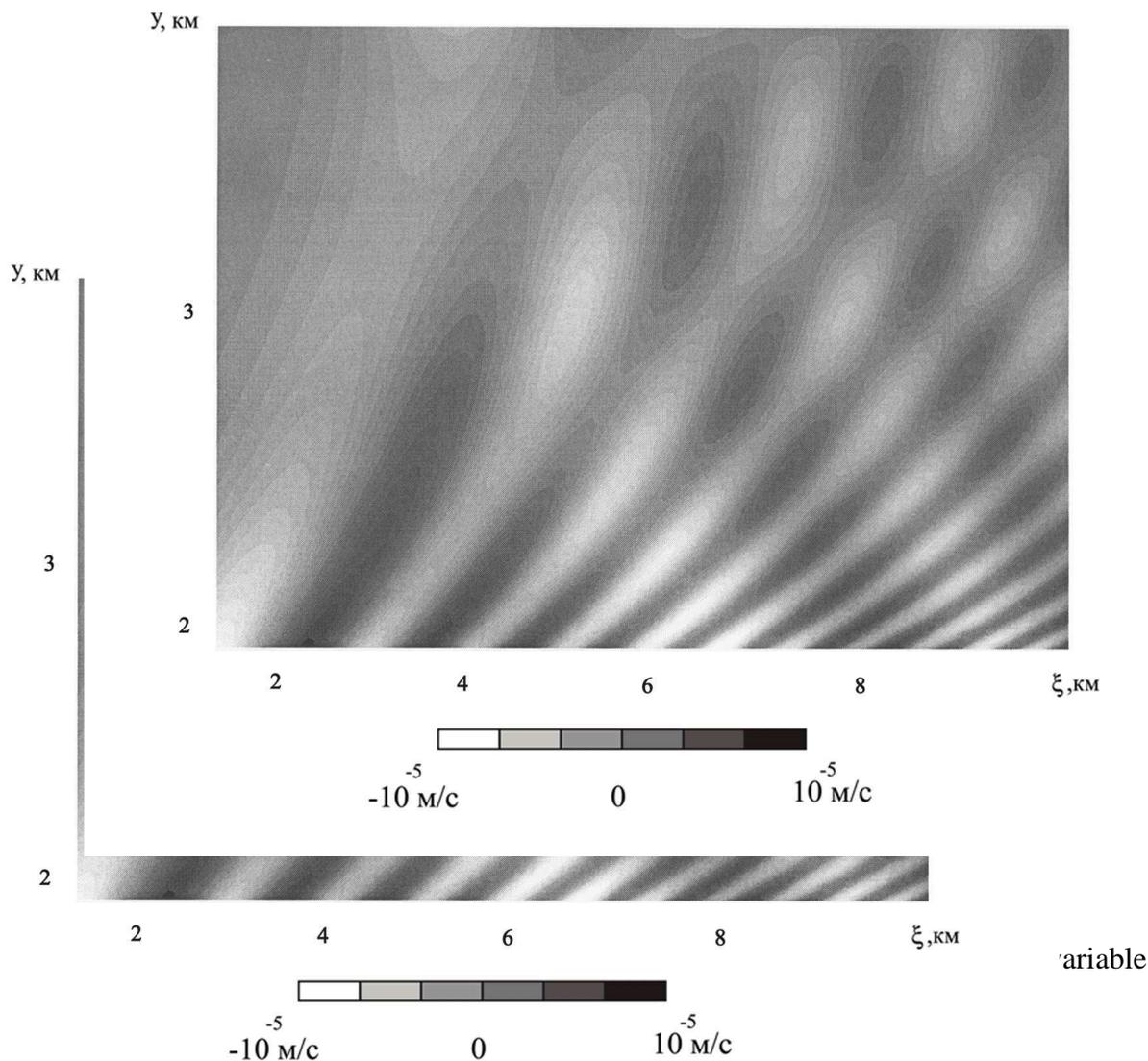


Fig. 2 Vertical component w of velocity from a source moving in stratified media of variable depth. in non-dimensional coordinates ($\xi = \xi/H, y = y/H$) plane.



APPLICATIONS

The results of this paper represent significant interest for physics and mathematics. Besides, asymptotic solutions, which are obtained in this paper, can be of significant importance for engineering applications, since the method of geometrical optics, which we modified in order to calculate the wave field near caustic, makes it possible to describe different wave fields in a rather wide class of other problems.

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