Internal Gravity Waves in Horizontally Inhomogeneous Ocean

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Introduction

The dynamics of wave motion in the ocean is currently of great interest because they are important in geophysics and oceanology. As a rule, theoretical analysis of these phenomena is based on the asymptotic methods, because the study of unperturbed hydrodynamic equations leads to asymptotic expansions (ansatzs, which is a German term for a type of solutions). These expansions permit solving the problems of perturbed equations, which can be used to describe the effects of nonlinearity, inhomogeneity, and non-stationary behavior of the real ocean. To obtain a detailed description of a wide range of physical phenomena related to wave dynamics of the stratified horizontally inhomogeneous unsteady ocean, it is necessary to start from the sufficiently developed mathematical models, which are usually quite complicated, nonlinear, and multi-parametric. They can be investigated completely only using efficient numerical methods. However, there are several cases, in which a preliminary qualitative concept of the phenomena under study can be obtained on the basis of simper asymptotic models and analytic methods for studying these models. These models then enter a set of "blocks" used to construct the complete pattern of wave dynamics, which permits discovering the correlation between different wave phenomena and their relationship. Sometimes, despite the seeming simplicity of the model assumptions, a successive choice of the solution form allows one to obtain physically interesting results [1, 2, 5, 14].

The propagation of internal gravity waves (IGW) in the ocean is strongly affected by the horizontal inhomogeneity and unsteady behavior of the basic

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hydrophysical parameters. In this contribution, we generalize a method of geometrical optics, i.e., the space-time ray method, which permits solving the problem of mathematical modeling of IGW dynamics in the horizontally inhomogeneous and vertically stratified ocean. The ray representations agree well with the intuitive and empirical concepts of IGW propagation in the real ocean. This method is sufficiently universal, and in many cases, this is the only possible method for approximate calculations of wave fields in the ocean. The most typical horizontal inhomogeneities of the real ocean are the variations in the bottom topography of the ocean, horizontal inhomogeneities of the density field, and unsteady ocean currents. An exact analytic solution can be obtained, for example, using the method of separation of variables only if the density distribution and the bottom topography can be described by sufficiently simple model functions. If the bottom topography and the ocean stratification are arbitrary, then one can construct only the asymptotic representations of the solution or solve the problem numerically. But the numerical solution does not permit obtaining and analyzing the qualitative characteristics of the wave field at large distances, which is necessary, for example, when solving the IGW detection problem by remote methods including, for example, radar imaging [8, 10, 12, 13, 15].

The mathematical modeling of IGW wave dynamics in the horizontally inhomogeneous and vertically stratified ocean is possible on the basis of a modified version of the space-time ray method (a method of geometrical optics). The specific form of asymptotic representations can be determined by solving the problems, which describe the IGW dynamics in the vertically stratified, horizontally homogeneous, and steady-state ocean. As a rule, when studying the evolution of IGW packets in the ocean with slowly varying and unsteady parameters, it is assumed that this wave packet is locally harmonic. In contrast to the majority of works, in which this problem has been studied, the proposed modified method of geometrical optics allows one to describe the structure of wave packets near singular surfaces such as caustics and wave fronts [3, 4, 6, 7].

The term “geometrical optics” has different meanings in the scientific literature. The geometrical optics understood in the narrow (or ray) sense deals only with the methods for constructing images by using the rays, while the geometrical optics understood in the wider (or wave) sense is a method for obtaining approximate descriptions of wave fields. In the wave interpretation, which is used in this paper, the rays, as a rule, form only the geometric skeleton, on which the wave filed is “sewn on”. According to the two previous interpretations of the geometrical optics, two periods in its development exist. The first ray period was ideologically completed by Hamilton’s fundamental works, which significantly influenced the development of the classical mechanics. The construction of rays underlies the instrumental optics, which is mainly oriented to design various optical devices. The contemporary wave period originates from the Debye’s works, which decisively influenced the formation of ray concepts in the wave theory [5].

The asymptotic representation of the solutions of wave packet propagation in the ocean with horizontally inhomogeneous density and numerical computations at the typical oceanic parameters testify that the horizontal inhomogeneity significantly affects the real IGW dynamics in the ocean. All results of wave dynamics modeling presented in this contribution can be used for arbitrary density distributions and other parameters of the stratified ocean. It is necessary to consider them in the context of consistency with the available data of IGW full-scale measurements in the ocean. Such methods for analyzing the wave fields are important not only because they are illustrative, universal, and efficient in various problems, but also because they can serve as a semi-empirical basis for the other approximate methods in the theory of wave packet propagation in the ocean.

The waves in media with slowly varying parameters have been studied in many publications, while the amount of works dealing with the problem of studying IGW in the media with variable parameters is quite rare (mainly because of significant mathematical difficulties encountered in these problems). In the first section of this paper, we present the basics of the space-time ray method (a method of geometrical optics) with regard to the special characteristics of IGW, which permits studying the wave dynamics in the horizontally inhomogeneous and vertically stratified ocean. In the second section, we discuss the problems of IGW propagation in the stratified ocean of variable depth.

IGW Fields in the Horizontally Inhomogeneous Ocean

Our analysis starts from a linear system of hydrodynamic equations [8, 10, 13, 15]

$$\begin{gathered}\rho_0\frac{\partial u_1}{\partial t}=-\frac{\partial p}{\partial x},\quad \rho_0\frac{\partial u_2}{\partial t}=-\frac{\partial p}{\partial y},\quad \rho_0\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+g\rho,\\ \frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial w}{\partial z}=0,\quad \frac{\partial \rho}{\partial t}+u_1\frac{\partial \rho_0}{\partial x}+u_2\frac{\partial \rho_0}{\partial y}+w\frac{\partial \rho_0}{\partial z}=0.\end{gathered} \tag{1.1}$$

Here (u_1, u_2, w) are components of the IGW velocity vector; p and ρ are perturbations of the pressure and density; g is the acceleration of gravity (the z axis is directed downwards). Using the Boussinesq approximation, which means that the unperturbed density $\rho_0(z,x,y)$ in the first three equations in system (1.1) is assumed to be constant, we reduce system (1.1) to the form:

$$\begin{gathered}\frac{\partial^4 w}{\partial z^2\partial t^2}+\Delta\frac{\partial^2 w}{\partial t^2}+\frac{g}{\rho_0}\Delta\left(u_1\frac{\partial \rho_0}{\partial x}+u_2\frac{\partial \rho_0}{\partial y}+w\frac{\partial \rho_0}{\partial z}\right)=0,\\ \frac{\partial}{\partial t}\left(\Delta u_1+\frac{\partial^2 w}{\partial z\partial x}\right)=0,\quad \frac{\partial}{\partial t}\left(\Delta u_2+\frac{\partial^2 w}{\partial z\partial y}\right)=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.\end{gathered} \tag{1.2}$$

We use the "rigid lid" condition at the surface and zero velocity at the bottom: $W=0$, $(z=0,\ -H)$, where H is the ocean depth as the boundary conditions. We assume that, in the media with horizontally inhomogeneous density field, the steady-state flows due to this field can be neglected. Indeed, it follows from the hydrodynamic equations that if the unperturbed density is a function of horizontal coordinates, then the existence of the steady-state density distribution $\rho_0(z,x,y)$ implies the existence of steady-state flows. These flows are rather slow, and they can be neglected in the first approximation. Therefore, it is usually assumed that

$\rho_0(z, x, y)$ is the background density field formed under the action of mass forces and non-adiabatic sources, and this field is given a priori, for example, by experimental data [2, 5].

Now we consider harmonic waves $(u_1, u_2, w) = \exp(i\omega t)(U_1, U_2, W)$. System (2) cannot be solved by the method of separation of variables, and therefore it is necessary to use asymptotic methods. The scales of horizontal variations in the ocean parameters can be greater than the scales of vertical variability [8, 10, 13, 15]. Further we introduce the dimensionless variables: $x^* = x/L$, $y^* = y/L$, $z^* = z/h$, where L is the characteristic scale of horizontal variations of density ρ_0 and h is the characteristic scale of vertical variations in ρ_0 (for example, the width of the thermocline). In the dimensionless coordinates, system (1.2) becomes (hereinafter, the asterisk in the indices is omitted)

$$\begin{gathered} -\omega^2\left(\tfrac{\partial^2 W}{\partial z^2} + \varepsilon^2 \Delta W\right) + \varepsilon^2 \tfrac{g_1}{\rho_0}\left(\varepsilon U_1 \tfrac{\partial \rho_0}{\partial x} + \varepsilon U_2 \tfrac{\partial \rho_0}{\partial y} + W \tfrac{\partial \rho_0}{\partial z}\right) = 0, \\ \varepsilon \Delta U_1 + \tfrac{\partial^2 W}{\partial z \partial x} = 0, \quad \varepsilon \Delta U_2 + \tfrac{\partial^2 W}{\partial z \partial y} = 0, \quad \varepsilon = \tfrac{h}{L} << 1, \quad g_1 = \tfrac{g}{h}. \end{gathered} \tag{1.3}$$

We seek for the asymptotic solution of (1.3) in the form typical for the method of geometrical optics [7].

$$\begin{gathered} \mathbf{V}(z, x, y) = \sum_{m=0}^{\infty} (i\varepsilon)^m \mathbf{V}_m(z, x, y) \exp(S(x, y)/i\varepsilon), \\ \mathbf{V}(z, x, y) = (U_1(z, x, y), U_2(z, x, y), W(z, x, y)), \end{gathered}$$

where function $S(x, y)$ and vector function $\mathbf{V}_m$, $m = 0, 1\ldots$, are sought. As a rule, below, we determine only the leading term of this asymptotic expansion for the vertical velocity component $W_0(z, x, y)$. We obtain the following from the two last equations in (1.3)

$$U_{10} = -\frac{i\partial S/\partial x}{|\nabla S|^2}\frac{\partial W_0}{\partial z}, \quad U_{20} = -\frac{i\partial S/\partial y}{|\nabla S|^2}\frac{\partial W_0}{\partial z}, \quad |\nabla S| = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2.$$

Equating the terms of order O(1), we obtain the equation for function $W_0(z, x, y)$. This equation is written as

$$\begin{gathered} \frac{\partial^2 W_0(z, x, y)}{\partial z^2} + |\nabla S|^2 \left(\frac{N^2(z, x, y)}{\omega^2} - 1\right) W_0(z, x, y) = 0, \\ W_0(0, x, y) = W_0(-H, x, y) = 0, \end{gathered} \tag{1.4}$$

where $N^2(z, x, y) = \frac{g_1}{\rho_0}\frac{\partial \rho_0}{\partial z}$ is the Brunt–Väisälä frequency depending on the vertical and horizontal coordinates. It is well known that the basic boundary-value vertical spectral problem for internal waves (1.4) has countably many eigenfunctions W_{0n} and eigenvalues $K_n(x, y, \omega) \equiv |\nabla S_n|$. Functions $W_{0n}(z, x, y)$ and $K_n(x, y, \omega)$ are assumed to be known; index n is omitted because we assume that all calculations are carried out for a separate wave mode. We use the eikonal equation $(\partial S/\partial x)^2 + (\partial S/\partial y)^2 = K^2(x,y)$ to determine function $S(x,y)$. In the plane case, the initial conditions for eikonal S are posed on line L: $x_0(\alpha), y_0(\alpha)$, $S(x,y)|_L = S_0(\alpha)$. To solve the eikonal equation, we construct the rays, i.e., the characteristics of this equation, which have the following form

$$\frac{dx}{d\sigma} = \frac{p}{K(x,y)}, \quad \frac{dp}{d\sigma} = \frac{\partial K(x,y)}{\partial x}, \quad \frac{dy}{d\sigma} = \frac{q}{K(x,y)}, \quad \frac{dq}{d\sigma} = \frac{\partial K(x,y)}{\partial y}, \tag{1.5}$$

where $p = \partial S/\partial x$, $q = \partial S/\partial y$, $d\sigma$ is the ray length element. The initial conditions of p_0 and q_0 for solution (1.5) are determined by solving the following system

$$p_0 \frac{\partial x_0}{\partial \alpha} + q_0 \frac{\partial y_0}{\partial \alpha} = \frac{\partial S_0}{\partial \alpha}, \quad p_0^2 + q_0^2 = K^2(x_0(\alpha), y_0(\alpha))$$

whose solution and the initial conditions $x_0(\alpha), y_0(\alpha), p_0(\alpha), q_0(\alpha)$ determine the ray $x = x(\sigma,\alpha)$, $y = y(\sigma,\alpha)$. After the rays are constructed, eikonal S can be determined by integrating along the ray: $S = S_0(\alpha) + \int_0^\sigma K(x(\sigma,\alpha), \; y(\sigma,\alpha))d\sigma$. Eigenfunction $W_0(z,x,y)$ is calculated up to multiplication by arbitrary function $A_0(x,y)$: $W_0(z,x,y) = A_0(x,y) f_0(z,x,y)$, where $f_0(z,x,y)$ is the solution of the basic vertical spectral problem with normalization $\int_0^H (N^2(z,x,y) - \omega^2) f_0^2(z,x,y)dz = 1$. Then, after rather cumbersome analytic calculations, we obtain the conservation law along the eikonal characteristics: $\frac{d}{d\sigma}\left(\ln \frac{A_0^2(x,y)I(x,y)}{K^2(x,y)}\right) = 0$, where $I(x,y)$ is the geometric divergence of the rays (characteristics). We note that the wave energy flux is proportional to $A_0^2 K^{-1} R$, where R is the width of an elementary ray tube; therefore, the quantity equal to the wave energy divided by the modulus of the wave vector is preserved in this case.

The long-range IGW fields in the real ocean are, as a rule, non-harmonic wave packets. Indeed, at a far distance form perturbation sources, the complete wave field is a sum of separate wave modes whose asymptotics, depending on the stratification, depth, and other parameters of the ocean, can be expressed in terms of the Airy function or the Fresnel integrals. Therefore, to study the problem of wave packet evolution in a horizontally smoothly inhomogeneous and unsteady stratified medium, it is necessary to use another ansatz [2, 5, 7].

We introduce slow variables $x^* = \varepsilon x, \quad y^* = \varepsilon y, \quad t^* = \varepsilon t$ (since z is not assumed to be a slow variable, we omit the asterisk in the index), where $\varepsilon = \lambda/L \ll 1$ is a small parameter characterizing the smoothness of the medium variations along the horizontal line (λ is the characteristic wave length, and L is the scale of horizontal inhomogeneity). Then system (1.2) for determining the velocity components (U_1, U_2, W) in these slow variables becomes

$$\frac{\partial^4 W}{\partial z^2 \partial t^2} + \varepsilon^2 \frac{\partial^2 W}{\partial t^2} + \frac{g}{\rho_0}\Delta\left(\varepsilon U_1 \frac{\partial \rho_0}{\partial x} + \varepsilon U_2 \frac{\partial \rho_0}{\partial y} + W\frac{\partial \rho_0}{\partial z}\right) = 0,$$
$$\varepsilon \Delta U_1 + \frac{\partial^2 W}{\partial z \partial x} = 0, \quad \varepsilon \Delta U_2 + \frac{\partial^2 W}{\partial z \partial y} = 0. \tag{1.6}$$

Further we consider the superposition of harmonic waves (in slow variables x, y, t) $W = \int \omega \sum_{m=0}^{\infty} (i\varepsilon)^m W_m(\omega, z, x, y) \exp\left(\frac{i}{\varepsilon}[\omega t - S_m(\omega, x, y)]\right) d\omega$, where functions $S_m(\omega, x, y)$ are assumed to be odd with respect to ω and $\min_{\omega} \partial S/\partial\omega$ is attained at $\omega = 0$ (for all x and y). We substitute this representation into (1.6) and see that function $W_m(\omega, z, x, y)$ for $\omega = 0$ has a pole of order m. Therefore, the model integrals, or phase functions $R_m(\sigma)$, for some terms of the asymptotic series are expressions $R_m(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i/\omega)^{m-1} \exp(i(\omega^3/3 - \sigma\omega))d\omega$, where the contour of integration bypasses point $\omega = 0$ from above, which ensures the exponential decay of functions $R_m(\sigma)$ for $\sigma \gg 1$. Functions $R_m(\sigma)$ have the following property $\frac{dR_m(\sigma)}{d\sigma} = R_{m-1}(\sigma)$, where $R_0(\sigma) = Ai'(\sigma)$, $R_1(\sigma) = Ai(\sigma)$, $R_2(\sigma) = \int_{-\infty}^{\sigma} Ai(u)du$, etc. Obviously, starting from the corresponding properties of the Airy integrals, we can conclude that functions $R_m(\sigma)$ are related as $R_{-1}(\sigma) + \sigma R_1(\sigma) = 0$, $R_{-3}(\sigma) + 2R_0(\sigma) - \sigma^2 R_1(\sigma) = 0$. For the model integrals $R_m(\sigma)$ describing the long-range IGW fields in the deep ocean, one can use the following expressions $R_0(\sigma) = \mathrm{Re}\int_0^{\infty} \exp(-it\sigma - it^2/2)\, dt \equiv \mathrm{Re}\Phi(\sigma)$; in this case, functions $R_m(\sigma)$ satisfy the recurrence relations $R_{-3}(\sigma) - 2iR_{-1}(\sigma) - i\sigma R_{-2}(\sigma) = 0$ and $R_{-1}(\sigma) + i\sigma R_0(\sigma) = 0$ [2, 5, 7]. It follows from the above and the structure of the first term of the uniform asymptotics (Airy or Fresnel wave) in a stratified and horizontally homogeneous medium that the solution of system (1.6) can be sought in the following form (index n is omitted for a separate wave mode)

$$W = \varepsilon^0 W_0(z, x, y, t)R_0(\sigma) + \varepsilon^a W_1(z, x, y, t)R_1(\sigma) + \varepsilon^{2a} W_2(z, x, y, t)R_2(\sigma) + \cdots,$$
$$\mathbf{U} = \varepsilon^{1-a}\mathbf{U}_0(z, x, y, t)R_1(\sigma) + \varepsilon\mathbf{U}_1(z, x, y, t)R_2(\sigma) + \varepsilon^{1+a}\mathbf{U}_2(z, x, y, t)R_3(\sigma) + \cdots,$$

where $\mathbf{U}$ is the vector of IGW horizontal velocity and the phase function argument $\sigma = (S(x, y, t)/a)^a \varepsilon^{-a}$ is assumed to be of the order of unity. This expansion agrees well with the general approach of the method of geometrical optics and the space-time ray method. Its generalization is used to study the dynamics of IGW fields in the horizontally inhomogeneous stratified ocean.

We also note that this structure of the solution implies that, in a horizontally inhomogeneous medium, the solution depends on both the "fast" (vertical coordinate) and "slow" (horizontal coordinates) variables. As a rule, the solution is sought in "slow" variables, and the structure elements depending on "fast" variables are obtained as integrals of some functions slowly varying along the space-time rays. This choice of the solution permits describing the uniform asymptotics of IGW fields propagating in the stratified ocean with slowly varying parameters, which is true both near and far from the wave fronts of a separate wave mode. If it is necessary to describe the behavior of the field only near the wave front, then one can use one of the methods of the geometrical optics, i.e., the “traveling wave” method, and the weakly dispersion approximation in the form of the corresponding local asymptotics to seek the representation of the phase function argument σ in the form $\sigma = \alpha(t, x, y)(S(t, x, y) - \varepsilon t)\varepsilon^{-a}$; here function $S(t, x, y)$ describes the wave front position. It is found by solving the eikonal equation $\nabla^2 S = c^{-2}(x, y, t)$, where $c(t, x, y)$ is the maximal IGW group velocity of the corresponding wave mode, i.e., the first term in the expansion of the dispersion curve at zero. Function $\alpha(t, x, y)$ (the second term of the dispersion curve expansion) describes the space-time evolution of the pulse width of non-harmonic Airy or Fresnel waves and can be found from some conservation laws along the eikonal equation characteristics whose specific form is determined by the physical conditions of the problems under study.

Further we compare the analytic results with the results of the analysis of measurements of IGW variability in a real medium with horizontally varying characteristics, namely, in the Northwest Pacific, according to the data recorded by moorings in the “Megapolygon” experiment in the Northwest Pacific. The measurements of the currents and the temperature recorded by the “Megapolygon” moorings allowed us to study the variability of tidal internal waves over the area of 460 × 520 km. The length of the tidal internal wave was calculated by integration of the basic IGW spectral equation with the real depth distribution of the Vaisala-Brunt frequency and with zero boundary conditions at the ocean surface and the ocean floor taking into account the Earth’s rotation. The wave length of the first mode in the “Megapolygon” area is equal approximately to 130 km, the wave length near the Emperor Ridge is greater (167 km), and it is equal to 156 km at a distance of 2000 km to the east. The wave propagation direction is also very stable and varies from 240° to 300°, which corresponds to the actual wave propagation to the west and northwest from the Emperor Ridge. Some diffraction of tidal internal waves was observed in the “Megapolygon” study site, i.e., the direction of wave propagation varied from the northwest in the southeast of the site and to the west in its northwest part [5].

Let us consider the amplitude variations of the internal tide in the course of its propagation to the west and to the east from the Emperor Ridge. The IGW amplitudes were calculated from the deviations of the temperature values measured on moorings; then, the values were divided by the average vertical gradient of temperature. Figure 1 illustrates the variations in the tidal internal wave amplitude versus distance. The calculations show that the IGW amplitude decreases

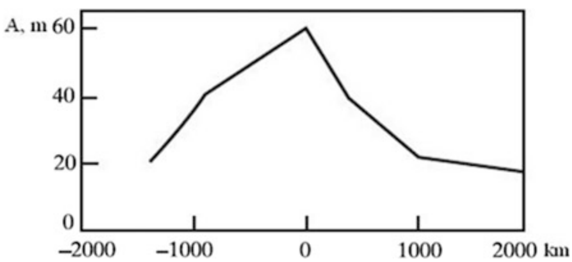

Fig. 1 The tidal IGW amplitude A versus the distance to the Emperor Seamounts

approximately by 10% at the distance equal to the length of the tidal internal wave (130–150 km) [11].

We can also estimate the influence of different factors, including the horizontal inhomogeneity of density, on the IGW decay. In the framework of the theory discussed above, we consider the evolution of IGW frequency ω corresponding to the semidiurnal period T = 12 h, which also admits slow variations in the stratification along the wave propagation path. The real geometry of the experiment allows us to assume that the problem under study is two-dimensional, which means that the stratification depends only on two variables: depth z and distance x along the wave propagation path.

Now we consider the case of constant depth H and stratification N linearly depending only on x: $N(x) = N_1 + (N_2 - N_1)\,x/L$, where L is the distance between the two observation points, $x = x_1 = 0$ is the initial point, $x = x_1 = L$ is the end point, and $N_{1,2} = N(x_{1,2})$. We consider only the first mode $\eta_1(z,x)$ of the amplitude of the vertical displacement of particles and omit its index. We seek the amplitude $\eta(z,x)$ in the form $\eta(z,x) = A(x)\,f(z,x)$, where $f(z,x)$ is the normalized eigenfunction of the standard boundary-value problem for the equation of internal waves with the normalization $\int_0^H (N^2(x) - \omega^2) f^2(z,x)\,dz = 1$, which has the form $f(z,x) = \sqrt{\frac{2}{H(N^2(x) - \omega^2)}}\sin(\pi z/H)$. Amplitude A(x) depending only on x is determined from the conservation law: $\frac{A^2(x_1)}{k^2(x_1)}da(x_1) = \frac{A^2(x_2)}{k^2(x_2)}da(x_2)$, where $k(x)$ is the absolute value of the horizontal wave vector, and $da(x)$ is the width of an elementary wave tube. Since the problem is two-dimensional, the width of the ray tube does not vary along the ray and the conservation law is simpler: $A(x)/k(x) = const$. Since we consider small values of ω, the velocity of wave propagation is close to the maximum group velocity $c(x) = N(x)\,H/\pi$; hence, the wave number is equal to $k(x) = \omega\pi/N(x)\,H$ and the corresponding wave length is equal to $\lambda(x) = 2N(x)\,H/\omega$. Then, under the assumption that the observation points are at the same depth, it follows from the conservation law $(A_{1,2} = A(x_{1,2}))$ that $A_1 N_1 = A_2 N_2$ or $A_2 = A_1\lambda_1/\lambda_2$. Then the total amplitude attains the following values $W_{1,2} = A_{1,2}\sqrt{\frac{2}{H\,(N_{1,2}^2 - \omega^2)}}$, which implies $W_2 = W_1 \frac{N_1}{N_2}\sqrt{\frac{(N_1^2 - \omega^2)}{(N_2^2 - \omega^2)}}$ or $W_2 = W_1 \lambda_1^2/\lambda_2^2$, because $\omega \ll N$, i.e., the amplitude of the internal gravity wave is inversely proportional to the squared wave length. The wave travel time τ along the horizontal ray is determined from the equation of characteristics $\frac{dx}{dt} = c(x)$, where $c(x) = (N_1 + ax)\,H/\pi$ and $a = (N_2 - N_1)/L$. Integrating this equation, we obtain the wave travel time $\tau = \frac{\pi}{aH}\ln\left(\frac{N_2}{N_1}\right) = \frac{T\,L}{(\lambda_2 - \lambda_1)}\ln\left(\frac{\lambda_2}{\lambda_1}\right)$. The available data of full-scale tests give the following values of the basic parameters of the problem: $\lambda_1 = 167$ km, $\lambda_2 = 156$ km, L = 2000 km. The wave attenuation coefficient without the wave length variations taken into account, which describes the amplitude decrease versus wave length denoted by β, gives the value of β: $\beta = 0.2^{167/2000} = 0.874$ with regard to relation $W_2\,/\,W_2 = 0.2 \equiv \beta^{t/T} = \beta^{L/\lambda}$ derived from the observation results. The attenuation with regard to the wave length variations along the ray, $W_2/W_1 = \beta^{\tau/T}$, with the theoretically calculated time of the wave travel time τ gives the following value $\beta = 0.878$. Thus, these estimates allow us to conclude that the influence of the density field inhomogeneities, which is taken into account in the above-described method for asymptotic representation of the wave fields, is one of the factors determining the scales of the space attenuation of IGW fields observed in field measurements.

Fields of IGW in the Ocean of Variable Depth

We consider one of the problems of IGW propagating in the stratified ocean of variable depth. In the framework of the linear theory, we study the non-viscous incompressible inhomogeneous medium with unperturbed density $\rho_0(z)$, which is bounded by surface $z = 0$ and ocean floor $z = \gamma y$ (the z axis is directed upwards, γ is the ocean floor slope). At point $x = x_0, y = y_0, z = z_0$ at the slope, there is a point mass source of power Q depending on time as $\exp(-i\omega t)$. The system of hydrodynamic equations for small perturbations of density ρ^*, pressure p^*, and velocity components (u_1, u_2, w) is written as [2–6]

$$\begin{gathered}\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial p^*}{\partial x}, \rho_0 \frac{\partial u_2}{\partial t} = -\frac{\partial p^*}{\partial y}, \rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p^*}{\partial z} + g\rho^* \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial w}{\partial z} = Q \exp(-i\omega t)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \\ \frac{\partial \rho^*}{\partial t} + w \frac{\partial \rho_0}{\partial z} = 0,\end{gathered} \quad (2.1)$$

where g is the acceleration of gravity. As the boundary conditions we pose the "rigid lid" condition at the ocean surface and zero mass flux at the ocean bottom

$$w = 0 \text{ at } z = 0, \; w + u_2 \gamma = 0 \text{ at } z = -\gamma y \quad (2.2)$$

Under the assumption that the time-dependence of all solutions is harmonic $(p^*, \rho^*, u_1, u_2, w) = \exp(-i\omega t)(p, \rho, U_1, U_2, W)$, we obtain the following system of equations with boundary conditions (2.2)

$$\begin{gathered}i\omega\rho_0 U_1 = \frac{\partial p}{\partial x}, \quad i\omega\rho_0 U_2 = \frac{\partial p}{\partial y}, \quad i\omega\rho_0 W = -\frac{c^2 \partial p}{\partial z}, \\ \frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial W}{\partial z} = Q\,\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad i\omega\rho = \frac{W \partial \rho_0}{\partial z},\end{gathered} \quad (2.3)$$

where $c^2 = \omega^2/(N^2 - \omega^2)$ and $N^2(z) = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$ is the Brunt-Väisälä frequency which is assumed constant: $N(z) = N = \text{const}$. These assumptions can be used to study the IGW fields in many regions of the World Ocean [12]. In the Boussinesq approximation, system (2.3) reduces to a single equation, for example, for pressure perturbations p with the corresponding boundary conditions

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial x^2}\right) = -i\omega Q\rho_0\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)/c^2, \qquad (2.4)$$

$$\frac{\partial p}{\partial z} = 0 \text{ at } z = 0, \; \frac{\partial p}{\partial z} - \frac{\gamma}{c^2}\frac{\partial p}{\partial y} = 0 \text{ at } z = -\gamma y. \qquad (2.5)$$

Since the variations in $\rho_0(z)$ are relatively small in the ocean, the value of ρ_0 in the right-hand side of (2.4) is understood, for example, as the value of the sea water density at the surface, i.e., we set $\rho_0 = \rho_0(0) = \text{const}$. Solution $p(x, y, z)$ must tend to zero as $\sqrt{x^2 + y^2 + z^2} \to \infty$. After function $p(x, y, z)$ is determined, velocity components (U_1, U_2, W) can be found from the first three equations of system (2.3), and density ρ is determined from the fifth equation in this system.

We change the variables as

$$y = r\text{ch}\varphi, \; z = -cr\text{sh}\varphi, \; r = \sqrt{y^2 - z^2/c^2}, \varphi = \frac{1}{2}\ln\frac{cy - z}{cy + z} \qquad (2.6)$$

We perform the Fourier transform with respect to variable x (without loss of generality, we can set $x_0 = 0$). Since the absolute value of the Jacobian of transition from the coordinates (y, z) to (r, φ) is equal to cr, problem (2.4), (2.5) implies the following plane boundary-value problem for the Fourier transform $P(r, \varphi, l)$ of function $p(r, \varphi, x)$

$$\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{r\partial r} - \frac{1}{r^2}\frac{\partial^2 P}{\partial \varphi^2} - l^2 P = \frac{q}{r}\delta(r - r_0)\delta(\varphi - \varphi_0), \qquad (2.7)$$

$$\frac{\partial P}{\partial \varphi} = 0 \quad \text{at} \quad \varphi = 0, \quad \frac{\partial P}{\partial \varphi} = 0 \quad \text{at} \quad \varphi = \varphi_r, \qquad (2.8)$$

$$r_0 = \sqrt{y_0^2 - z_0^2/c^2}, \quad \varphi_0 = \frac{1}{2}\ln\frac{c\,y_0 - z_0}{c\,y_0 + z_0}, \quad \varphi_r = \frac{1}{2}\ln\frac{c+\gamma}{c-\gamma}, \quad q = i\omega Q\rho_0/c. \qquad (2.9)$$

The solution of three-dimensional boundary-value problem (2.4), (2.5) with respect to variables (r, φ, x) is obtained from the solution of the plane problem (2.7), (2.8) by using the inverse Fourier transform

$$p(r, \varphi, x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(r, \varphi, l)\exp(ilx)dl. \qquad (2.10)$$

We assume that the ocean floor slope γ is less than c or, in the trigonometric terminology, we assume that the ocean bottom slope is subcritical (the critical slope is $\gamma = c$) [3–6].

The homogeneous Eq. (2.7) with zero right part has real solutions $P(r, \varphi, l) = K_{i\mu}(lr)\cos(\mu\varphi)$ decreasing at infinity, where μ is an arbitrary real number and $K_{i\mu}(lr)$ is the Macdonald function with imaginary index satisfying the modified parametric Bessel equation $LK_{i\mu}(lr) = 0$, where $L = r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{r\partial r} + (\mu^2 - r^2 l^2)$. We note that function $K_{i\mu}(lr)$ is real if the values of μ are real and argument lr is positive. Hence, we write the delta function $\delta(r - r_0)$ using a pair of direct and inverse Kantorovich-Lebedev transformations [5, 6]

$$F(\mu) = \int_0^{+\infty} K_{i\mu}(x)\frac{f(x)}{x}dx, f(x) = \frac{2}{\pi^2}\int_0^{+\infty} \mathrm{sh}(\pi\mu)K_{i\mu}(x)F(\mu)\mu d\mu\,.$$

This implies the expansion of the delta function (completeness condition) in the form

$$\delta(r - r_0) = \frac{2}{r\pi^2}\int_0^{+\infty} \mathrm{sh}(\pi\mu)K_{i\mu}(lr)\,K_{i\mu}(lr_0)\mu d\mu. \quad (2.11)$$

We seek the solution of problem (2.7) in the form

$$P(r, \varphi, l) = \frac{2q}{\pi^2}\int_0^{+\infty} \mathrm{sh}(\pi\mu)K_{i\mu}(lr)\,K_{i\mu}(lr_0)\Phi_\mu(\mu)\mu d\mu, \quad (2.12)$$

where the function of the angular variable $\Phi_\mu(\varphi)$ is still unknown. Substituting (2.11) and (2.12) into (2.7), we obtain the boundary-value problem for determining this function

$$\begin{gathered} \frac{d^2\Phi_\mu(\varphi)}{d\varphi^2} + \mu^2\Phi_\mu(\varphi) = -\delta(r - r_0), \\ \frac{d\Phi_\mu(0)}{d\varphi} = \frac{d\Phi_\mu(\varphi_r)}{d\varphi} = 0. \end{gathered} \quad (2.13)$$

It follows from (2.13) that $\Phi_\mu(\varphi)$ is the angular Green function of the form

$$\Phi_\mu(\varphi) = -\frac{1}{\mu^2\varphi_r} - \frac{2}{\varphi_r}\sum_{n=1}^{\infty}\frac{\cos(\varphi\mu_n)\,\cos(\varphi_0\mu_n)}{\mu^2 - \mu_n^2}, \quad \mu_n = 2\pi n/\ln\left(\frac{c+\gamma}{c-\gamma}\right), \quad n \geq 1. \quad (2.14)$$

In the expression for $P(r, \varphi, l)$ in (2.12), we consider a single wave mode $(n \geq 1)$

$$P_n(r, \varphi, l) = -\frac{4q\,\cos(\varphi\mu_n)\,\cos(\varphi_0\mu_n)}{\varphi_r\pi^2}\int_0^{+\infty}\frac{\mathrm{sh}(\pi\mu)\,K_{i\mu}(lr)K_{i\mu}(lr_0)\mu d\mu}{\mu^2 - \mu_n^2}. \quad (2.15)$$

Here, the integral is understood in the sense of the principal value. Formula (2.15) can also be used for $n = 0$ if we set $\mu_0 = 0$ and decrease the coefficient of the integral by a factor of two. First, we consider the case $r > r_0$. To deform the contour of integration over μ in expression (15), we use formula $K_\nu(t) = \pi(I_{-\nu}(t) - I_\nu(t))/(2\sin(\pi\mu))$ which, in this case with $\nu = i\mu$ for function $K_{i\mu}(lr_0)$, becomes

$$K_{i\mu}(lr_0) = -\pi \mathrm{Im}\,(I_{i\mu}(lr_0))/\mathrm{sh}(\pi\mu), \tag{2.16}$$

because functions $I_{i\mu}(x)$ and $I_{i\mu}(-x)$ are complex conjugate function. The intergrand in (2.15) is even with respect to μ; hence, we can use (2.16) to obtain

$$P_n(r,\varphi,l) = \frac{2q\cos(\varphi\mu_n)\cos(\varphi_0\mu_n)}{\pi\varphi_r}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{K_{i\mu}(lr)I_{i\mu}(lr_0)\mu d\mu}{\mu^2-\mu_n^2}. \tag{2.17}$$

Now the contour of integration in (2.17) can be closed in the lower half-plane. To verify this, we use the asymptotic expansions of $K_{i\mu}(x)$ and $I_{i\mu}(x)$ for $\mu = -i\nu$ as $\nu \to \infty$: $K_\nu(lr) \approx \sqrt{\pi/2\nu}(2\nu/elr)^\nu$, $I_\nu(lr_0) \approx \sqrt{\pi/2\nu}(2\nu/er_0 l)^\nu/2\sqrt{2}$. Then we can obtain $K_\nu(lr)I_\nu(lr_0) \approx \pi\,\exp\,(-\nu(\ln r - \ln r_0))/4\nu\sqrt{2}$. This implies that the integrand is exponentially small in the lower half-plane for $r > r_0$. Then, taking into account the residues at points $\mu = \pm\mu_n$, we have

$$P_n(r,\varphi,l) = -\frac{2q\cos(\varphi\mu_n)\cos(\varphi_0\mu_n)}{\varphi_r}\mathrm{Re}(K_{i\mu_n}(lr)I_{i\mu_n}(lr_0)). \tag{2.18}$$

In the case $r < r_0$, we represent function $K_{i\mu}(lr)$ in the form (2.16) and closing the contour of integration in the lower half-plane we obtain expression (2.18), where it is necessary to interchange r and r_0. These expressions can be written as a single expression if we introduce notations $r_- = \min(r, r_0)$, $r_+ = \max(r, r_0)$

$$P_n(r,\varphi,l) = -\frac{2q\cos(\varphi\mu_n)\cos(\varphi_0\mu_n)}{\varphi_r}\mathrm{Re}(K_{i\mu_n}(lr_+)I_{i\mu_n}(lr_-)). \tag{2.19}$$

In the case $n = 0$, we similarly have

$$P_0(r,\varphi,l) = -\frac{q}{\varphi_r}\mathrm{Re}\,(K_0(lr_+)I_0(lr_-)). \tag{2.20}$$

Now we calculate the inverse Fourier transform (2.10) for the n-th mode $(n \geq 0)$ with regard to the fact that the steady-state standing wave is an odd function of variable x; as a result, we obtain $p_n(r,\varphi,x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} P_n(r,\varphi,l)\cos(lx)dl$. This integral can be expressed in the terms of the hypergeometric function

$$p_n(r,\varphi,x) = -\frac{q\varepsilon_n\cos(\varphi\mu_n)\cos(\varphi_0\mu_n)}{\sqrt{\pi r\, r_0\varphi_r}}\mathrm{Re}Z,$$
$$Z = \frac{\Gamma(i\mu_n+1/2)}{\Gamma(i\mu_n+1)}(\tau/2)^{i\mu_n+1/2}F\left(\frac{i\mu_n+1/2}{2},\frac{i\mu_n+3/2}{2},i\mu_n+1,\tau^2\right), \quad (2.21)$$

where $\Gamma(z)$ is the gamma function, $F(\alpha,\beta,\gamma,z)$ is the hypergeometric function, $\tau = 2r\, r_0/(r^2 + r_0^2 + x^2)$, $\varepsilon_n = 1/2$ for $n=0$, and $\varepsilon_n = 1$ for $n \geq 1$. The complete solution is obtained as a sum of all modes: $p(r,\varphi,x) = \sum_{n=0}^{\infty} p_n(r,\varphi,x)$, where r and φ are determined from (2.6), and r_0, φ_0, φ_r are determined from (2.9). We note that small values of τ correspond to the far field distance from the perturbation source, i.e., to the large values of r and x; a separate mode $p_n(r,\varphi,x)$ can be approximated by the expansion of the hypergeometric function in a series for $0 \leq z < 1$

$$F(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}z^2 + \cdots, \quad (2.22)$$

where, $\alpha = \frac{i\mu_n+1/2}{2}$, $\beta = \frac{i\mu_n+3/2}{2}$, and $\gamma = i\mu_n + 1$. However, as the mode number n increases at fixed z, it is required to take even greater number of terms in expansion (2.22) (the number of terms is $m \approx \mu_n z$), which hampers the calculation of wave modes with large numbers. For the further summation of the series (2.22), we use the WKB asymptotics of the hypergeometric function in (2.21)

$$F(\tau^2) \approx \exp\left(-\frac{i\mu_n}{2}\left(\ln\frac{\tau^2}{4} + \ln\frac{1+\sqrt{1-\tau^2}}{1-\sqrt{1-\tau^2}}\right)\right)/\sqrt{41-\tau^2}. \quad (2.23)$$

We use the asymptotics of the gamma function in (2.21) for large values of μ_n: $\frac{\Gamma(i\mu_n+1/2)}{\Gamma(i\mu_n+1)} \approx \exp(-i\pi/4)/\sqrt{\mu_n}$. Finally, we obtain the following expression for the WKB asymptotics of a separate wave mode at large μ_n

$$p_n(r,\varphi,x) \approx -\frac{q\sqrt{\tau}\cos(\varphi\mu_n)\cos(\varphi_0\mu_n)}{\sqrt{2\mu_n\pi r\, r_0\, \varphi_r}}\cos\left(\frac{\mu_n}{2}\ln\frac{1+\sqrt{1-\tau^2}}{1-\sqrt{1-\tau^2}} + \pi/4\right). \quad (2.24)$$

It is interesting to note that if we formally set $\mu_n \to \infty$ in expansion (2.22), let $z \to 0$ in the WKB asymptotics (2.23) for $F(z)$, and take into account that $z\mu_n \approx O(1)$, then, in both cases, we obtain the same value equal to $\exp(-iz\mu_n/4)$. Thus, expansion (2.22) and WKB asymptotics (2.23) are mutually consistent, i.e., there is a domain of z, μ_n, where these expressions coincide. It follows from (2.24) that the amplitude of the n-th mode decreases as $((x^2+y^2)n)^{-1/2}$ for large x, y. Expanding the phase in (2.24) for small τ, we see that, for large y, the half-wave length along axis y increases as $\pi y/\mu_n$, and along axis x, as $\pi x/2\mu_n$. The numerical calculations with the real parameters of the ocean show that the exact and asymptotic solutions agree well, except for the immediate vicinity of the perturbation source, where the argument of the hypergeometric function tends to unity, which follows from the construction of the asymptotic solution. We note that expression (2.24) formally requires that $\mu_n \to \infty$, but already for the first mode $n=1$, asymptotic formula (2.24) gives a qualitatively true description of the exact solutions. The asymptotics of the zero mode can be calculated from (2.21) by setting $\mu_n=0$. Then, taking into account that $F\left(\frac{1}{4}, \frac{3}{4}, 1, \tau^2\right) = \frac{2}{\pi\sqrt{1+\tau}} K\left(\frac{2\tau}{1+\tau}\right)$, where $K(x) = \int_0^{\pi/2} \left(1-(x \sin \varphi)^2\right)^{-1/2} d\varphi$ is an elliptic integral of the first kind, we obtain the following expression

$$p_0(r,\varphi,x) \approx -\frac{q\sqrt{\tau}}{\pi\sqrt{1+\tau}\sqrt{2r\,r_0}\varphi_r} K\left(\frac{2\tau}{1+\tau}\right). \tag{2.25}$$

We use the asymptotics of $K(x)$ as $x \to 1$ with the leading term $K(x) \approx \ln 4 - \ln(1-x)/2$ and finally obtain the expression for the asymptotics of the zero mode

$$p_0(r,\varphi,x) \approx -\frac{q\sqrt{\tau}}{\pi\sqrt{1+\tau}\,\sqrt{2r\,r_0\varphi_r}} \left(\ln\left(\frac{1-\tau}{1+\tau}\right)/2\right). \tag{2.26}$$

We note that the exact and asymptotic solutions coincide completely near the perturbation source. There is difference between them at far distances from the source. This is related to the fact that the asymptotics of the elliptic integral works well as the argument tends to unity. Nevertheless, in the far region, the asymptotics qualitatively true describes the exact solution with an error at most equal to a few percent. The obtained asymptotic representations of the solutions for separate wave modes, including the zero mode, permit calculating the complete wave field. The sum of asymptotics (2.24) of infinitely many wave modes $(n=1,2,\ldots.)$ is expressed in terms of semi-logarithmic function

$$Li_{1/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}, \quad B_{\pm}^{\pm} = \exp(i\pi(\pm\varphi\pm\varphi_0 + A(\tau))/\varphi_r), \quad A(\tau) = \frac{1}{2}\ln\frac{1-\sqrt{1-\tau^2}}{1+\sqrt{1-\tau^2}},$$
$$\sum_{n=1}^{\infty} p_n(r,\varphi,x) = -\frac{q\sqrt{\tau}\,\exp(-i\pi/4)}{8\pi\sqrt{4}1-\tau^2\sqrt{rr_0}\,\varphi_r}\left(Li_{1/2}(B_+^+) + Li_{1/2}(B_-^+) + Li_{1/2}(B_+^-) + Li_{1/2}(B_-^-)\right). \tag{2.27}$$

The complete wave field is the real part of expression (2.27) and the zero mode (2.25). The semi-logarithmic function in (2.27) becomes infinite at the points, at which the condition $\pi(\pm\varphi \pm \varphi_0 + A(\tau))/\varphi_0 = 2\pi m, m = 0,1,2,\ldots$ is satisfied. The locus of points (x,y,z) satisfying this condition determines a system of rays if one of the variables is fixed. On planes (y,z) and (x,z), these solutions determine a pair of ascending rays and a pair of descending rays, which are radiated from the source and then reflect from the sloping ocean floor. Figure 2 presents the shadow picture of the complete wave field (level lines) on plane (y,z) for $x=40$ m; the other

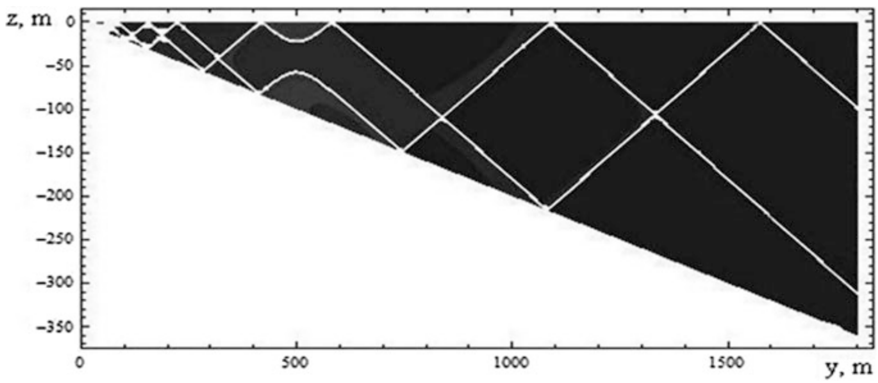

Fig. 2 Amplitude structure of IGW (pressure) in stratified ocean with non-uniform depth: analytical results

computational parameters are typical for the real ocean parameters: $N = 0.001\,\mathrm{s}^{-1}$, $\omega = 0.004\,\mathrm{s}^{-1}$, $\gamma = 0.2$, $c = 0.44$, $\rho_0 = 1000\,\mathrm{kg/m}^{-3}$, $Q = 1600\,\mathrm{m}^3/\mathrm{s}$, $y_0 = 500\,\mathrm{m}$, $z_0 = -4\,\mathrm{m}$.

These results clearly illustrate the ray structure of the constructed solutions, in particular, the set of incident and reflected rays; moreover, the cotangent of the angle between the incident ray and the vertical is approximately equal to 0.44, which agrees well with the ray theory. Indeed, according to this theory, the direction of group velocity Θ and the energy propagation direction are determined by expression $\mathrm{ctg}^2\Theta = c^2 = \omega^2/(N^2 - \omega^2)^2$ [8, 10, 13, 15]. The solutions are singular on the rays, because the model of ideal medium is used. The main contribution to the singularity is given by infinitely many short-wave modes with large numbers. In reality, to obtain the complete wave field, it is necessary to consider finitely many modes. This number is approximately determined by the Stokes characteristic scale $D = \sqrt{2\nu_0/N}$, where ν_0 is the kinematic viscosity and N is the Brunt-Väisälä frequency. Obviously, the wave modes with large numbers whose wave length is less that D do not contribute to the solution.

For comparison with the analytic results, in Fig. 3, we show the results of numerical simulation of the complete system of hydrodynamic equations, which describes the evolution of nonlinear wave perturbations over uneven ocean floor (Bay of Biscay, more than 60 wave modes were summed) [9].

The results show that the ray structure of the solution (Fig. 2) is clearly identified and, as the estimates show, the amplitude-phase structure of the wave fields is quite well described by asymptotic formulas (2.27).

Figure 4 illustrates the results of full-scale measurements of the amplitude structures of the tidal IGW in the same region of the World Ocean [9]. These full-scale data show that the wave patterns with profound ray structure can actually be observed in the real ocean, especially, when the IGW evolution over uneven ocean floor is investigated. In particular, the analytic, numerical, and full-scale data

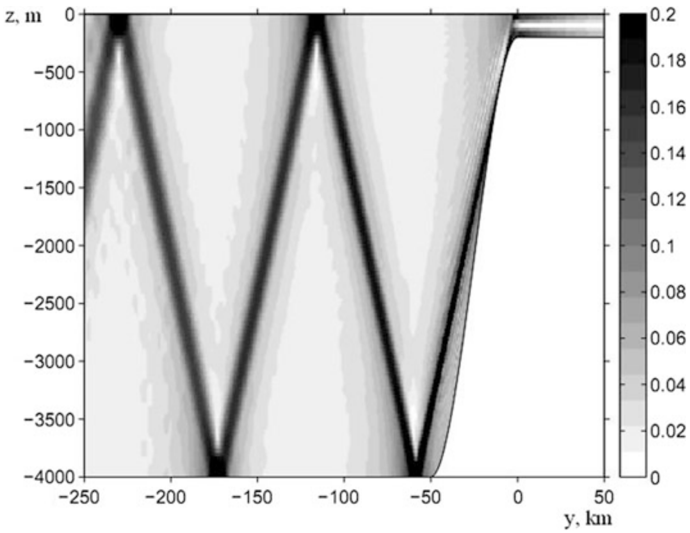

Fig. 3 Amplitude structure of IGW (velocity, m/s^{-1}) in stratified ocean with non-uniform depth: numerical simulation

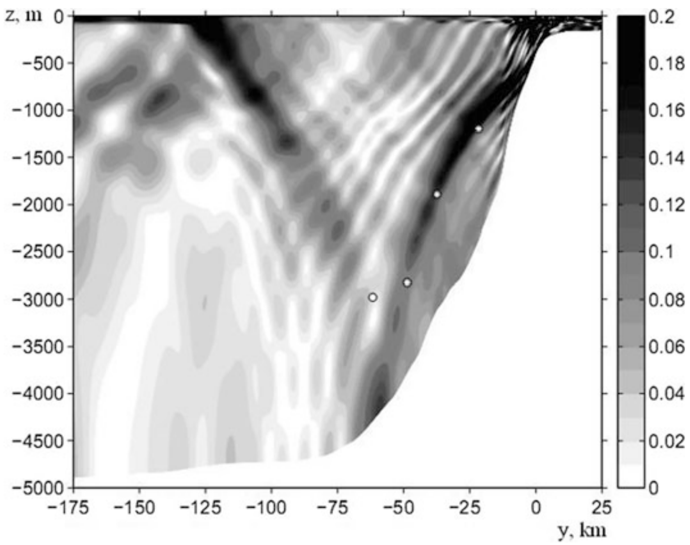

Fig. 4 Amplitude structure of IGW (velocity, m/s^{-1}) in stratified ocean with non-uniform depth: measurements results

show that the width of wave beams decreases as the shore is approached. Formally, in the linear statement, the width of the reflected IGW beam can be arbitrarily small for appropriate relations between the medium parameters (stratification, the ocean floor slope angle); hence, a significant local intensification of waves occurs near the ocean shore. It is clear that in the real ocean, the wave field energy remains finite in such spatial domains due to the action of nonlinear mechanisms of dissipation and turbulent mixing [1].

Conclusions

Thus, in the first section of the paper, a general method for calculating IGW fields in the horizontally inhomogeneous ocean is outlined, namely,

- for an arbitrary distribution of the Brunt-Väisälä frequency, the basic vertical spectral IGW problem is solved and the corresponding normalized eigenfunctions and eigenvalues are determined;
- the characteristic systems with appropriate initial conditions are solved numerically;
- after the characteristics (rays) are calculated, the eikonal (phase value) of the phase functions is determined by numerical integration along these rays;
- the geometric divergence of the ray tubes is determined, for example, by numerical differentiation of closely located characteristics;
- the IGW amplitude is calculated from the equations of the corresponding conservation laws along the rays (characteristics), in which the right parts of the relations are determined by using the locality principle, i.e., it is assumed that the ocean parameters remain horizontally unchanged over specific spatial intervals. Thus, it is assumed that the ocean is horizontally homogeneous on these space-time scales, and its density arbitrarily depends on the vertical coordinate.

The solutions obtained in the second section of the paper are exact and exhibit typical ray pattern of the IGW fields in the stratified ocean of variable depth obtained without using the mathematical methods of geometrical optics.

The universal character of the proposed asymptotic methods of modeling IGW fields in the ocean allows us to efficiently calculate the wave fields and, in addition, analyze qualitatively the solutions. This opens wide opportunities for investigating the wave fields in general, which is also important for formulating correct statements of mathematical models of wave dynamics and for obtaining express evaluations in the field measurements of internal waves.

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