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FAR FIELDS OF INTERNAL WAVES EXCITED BY A PULSING SOURCE IN A STRATIFIED MEDIUM WITH SHEAR FLOWS

V. V. Bulatov* and Yu. V. Vladimirov

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Abstract: A problem of the far field of internal gravity waves excited by an oscillating point source of perturbations in a stratified medium with a shear flow is solved. A model distribution of the shear flow velocity by depth is considered and an analytical solution to this problem is obtained in the form of the characteristic Green function expressed in terms of the modified Bessel functions of the imaginary index. Expressions for dispersion relations are obtained and integral representations of solutions are constructed. The dependences of the wave characteristics of the excited fields on the main parameters of the used stratification models, flows, and generation regimes are investigated.

Keywords: stratified medium, internal gravity waves, buoyancy frequency, shear flows.

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INTRODUCTION

Among the wide variety of wave processes observed in various physical media, a special place is occupied by the interaction of excited waves with hydrodynamic flows [1–3]. The motion of a stratified medium is one of the main factors influencing the dynamics of internal gravity waves (IGWs) both in natural conditions (ocean and Earth's atmosphere) and in engineering devices. Modern studies devoted to IGW dynamics in natural stratified media with account for the presence of flows are carried out using asymptotic methods for investigating analytical wave generation models. In the linear approximation, the existing approaches to the description of the wave pattern of excited fields of IGWs are based on the representation of wave fields as Fourier integrals and their asymptotic analysis [2–10]. Under real oceanic conditions, it is necessary to consider IGWs propagating along with mean flows with a vertical velocity shift, and the vertical velocity can vary by tens of centimeters or meters per second, i.e., it has the same order as the maximum velocity of IGWs. Such flows have a significant impact on the IGW propagation. If the horizontal variation rate of the flows is much higher than the IGW velocity and the time variation rate of the mean flow is much greater than the periods of internal waves, then the most adequate mathematical model is the model of stationary and horizontal homogeneous mean shear flows [1, 5–8].

The goal of this study is to investigate the dynamics of far fields of IGWs in stratified media with account for shear flows.

Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119526 Russia; *internalwave@mail.ru; vladimyura@yandex.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 60, No. 6, pp. 45–52, November–December, 2019. Original article submitted March 21, 2019; revision submitted May 21, 2019; accepted for publication May 27, 2019.

*Corresponding author.

1. FORMULATION OF THE PROBLEM

A layer of vertically stratified fluid of thickness H is considered. Let $(U(z), V(z))$ be the shear flow velocity vector on horizon z. The object of further analysis is a linearized system of hydrodynamic equations with respect to an unperturbed state, which has the form [1–4, 8]

$$\rho_0 \frac{DU_1}{Dt} + \frac{\partial p}{\partial x} = 0, \qquad \rho_0 \frac{DU_2}{Dt} + \frac{\partial p}{\partial y} = 0, \qquad \rho_0 \frac{DW}{Dt} + \frac{\partial p}{\partial z} + \rho g = 0,$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial W}{\partial z} = 0, \qquad \frac{\partial \rho}{\partial t} + W \frac{\partial \rho_0}{\partial z} = 0,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} + V(z) \frac{\partial}{\partial y},$$

where U_1, U_2, and W are the perturbation velocity components, p and ρ are the pressure and density perturbations, and $\rho_0(z)$ is the unperturbed density of the medium. The Boussinesq approximation is used to obtain the equation for the vertical velocity component [1, 4]

$$\frac{D^2}{Dt^2} \Delta W - \frac{D}{Dt} \Big(\frac{d^2 U}{dz^2} \frac{\partial W}{\partial x} + \frac{d^2 V}{dz^2} \frac{\partial W}{\partial y} \Big) + N^2(z) \Delta_2 W = Q(t, x, y, z),$$

$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \qquad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad N^2(z) = -\frac{g}{\rho_0(z)} \frac{d\rho_0(z)}{dz}, \tag{1.1}$$

where $N(z)$ is the Brunt–Väisälä frequency (buoyancy frequency) and g is the acceleration of gravity. The form of the function $Q(t, x, y, z, z_0)$ depends on the nature of a perturbation source. In the case with a force aimed in a vertical direction, we have $Q(t, x, y, z, z_0) = \delta'(t)\delta(z - z_0)(\delta''(x)\delta(y) + \delta''(y)\delta(x))$. In the case with the point mass source under consideration, we have $Q(t, x, y, z, z_0) = \delta''(t)\delta(x)\delta(y)\delta'(z - z_0)$. In this paper, this function has the form $Q(t, x, y, z, z_0) = \delta(x)\delta(y)\delta(z - z_0)\, e^{i\omega t}$, i.e., the Green function for an oscillating point source of perturbations, located at depth z_0, is considered. Obviously, due to the linearity of the problem under consideration, the resulting asymptotic solutions can be used to determine expressions for the fields of the IGWs generated by arbitrary nonlocal sources [2–4].

Atmospheric cyclones are one of the main sources of IGW generation in the ocean. Wave fields excited by this generation mechanism can play a significant role in various types of energy transfer in the ocean. The experimental detection of a trace of IGWs excited by a moving hurricane is a significant achievement of modern oceanology. At large distances, real perturbation sources (typhoon, atmospheric pressure perturbations, and cyclone) allow for physically justified approximation with the help of a system of localized point sources taken with specific weights. This approach is generally accepted and physically justified for solving most problems of simulation of the IGW dynamics in the ocean with account for shear flows. At large distances from perturbation sources, the source shape practically does not affect the wave characteristics of the IGW and is almost completely determined by the stratified medium parameters and the corresponding dispersion laws [1, 4, 6, 7, 9, 10].

Boundary conditions are specified in the form (the vertical axis of z is directed upward)

$$z = 0, \quad z = -H: \qquad W = 0. \tag{1.2}$$

Next, the following assumptions are used. The Brunt–Väisälä frequency is regarded as constant: $N(z) = N = \text{const}$; the flow is considered to be one-dimensional: $V(z) \equiv 0$. The function $U(z)$ is a linear depth function $U(z) = U_0 + (U_0 - U_H)z/H$, where $U_0 = U(0)$ and $U_H = U(-H)$. This hydrological model is widely used in real oceanological calculations and makes it possible to describe the main properties of wave dynamics with allowance for real changes in the density of the marine environment, observed in the field measurements of IGWs in the oceans, and investigate the problem analytically [1, 6, 7, 11, 12]. Then, we use the dimensionless coordinates and variables

$$x^* = \frac{\pi x}{H}, \quad y^* = \frac{\pi y}{H}, \quad z^* = \frac{\pi z}{H}, \quad W^* = \frac{WHN^2}{\pi q}, \quad \omega^* = \frac{\omega}{N}, \quad t^* = tN,$$

$$M(z^*) = \frac{\pi U(z^*)}{NH} = a + bz^*, \qquad a = \frac{\pi U_0}{NH}, \qquad b = \frac{U_0 - U_H}{NH}$$

(the superscript asterisk is omitted below) along with Eqs. (1.1) and (1.2) to obtain

$$\Bigl(\frac{\partial}{\partial x} + M(z)\,\frac{\partial}{\partial x}\Bigr)^2 \Delta W + \Delta_2 W = \mathrm{e}^{i\omega t}\,\delta(x)\delta(y)\delta(z - z_0); \tag{1.3}$$

$$z = 0, \quad z = -\pi: \qquad W = 0. \tag{1.4}$$

Next, the following parameter values are used: $a = 0.8$, $b = 0.2$, and $\omega = 0.54$. The function $W(t, x, y, z)$ is written as $W(t, x, y, z) = \mathrm{e}^{i\omega t}\, w(x, y, z)$. The solution (1.3), (1.4) is sought for in the form of the Fourier integrals

$$w(x, y, z) = \frac{1}{4\pi^2} \int\limits_{-\infty}^{\infty} d\nu \int\limits_{-\infty}^{\infty} \varphi(\mu, \nu, z)\,\mathrm{e}^{-i(\mu x + \nu y)}\; d\mu. \tag{1.5}$$

The function $\varphi(\mu, \nu, z)$ can be determined by solving the boundary-value problem

$$\begin{gathered} \frac{\partial^2 \varphi(\mu, \nu, z)}{\partial z^2} + k^2((\omega - \mu M(z))^{-2} - 1)\varphi(\mu, \nu, z) = -\delta(z - z_0)(\omega - \mu M(z))^{-2}, \\ \varphi(\mu, \nu, 0) = \varphi(\mu, \nu, -\pi) = 0, \qquad k^2 = \mu^2 + \nu^2. \end{gathered} \tag{1.6}$$

2. CONSTRUCTION OF ANALYTICAL SOLUTIONS

The role of two linearly dependent solutions of the problem (1.6) with a zero right side is played by solutions expressed via a modified Bessel function with an imaginary index [13]:

$$f_{1,2}(z) = \sqrt{2\beta(\omega - \mu M(z))}\; I_{\pm i\lambda}(\beta(\omega - \mu M(z))).$$

Here the subscripts 1 and 2 correspond to the plus and minus signs, respectively, $\lambda = \sqrt{\beta^2 - 1/4}$, and $\beta = k/(b\mu)$. The functions $f_1(z)$ and $f_2(z)$ are complex conjugate. It is assumed that the shear flow velocity is positive over the entire depth of the stratified medium, i.e., $a > 0$ and $a - b\pi > 0$. It is also regarded that, for the Richardson number, the Miles stability condition $\mathrm{Ri} = N^2(\partial U/\partial z)^{-2} > 1/4$ is fulfilled, i.e., $b^2 < 4$ [1–4, 8]. Hence, $\beta^2 > 1/4$ and the values of λ are real. The function $I_{i\lambda}(\tau)$ oscillates for the real values of λ and for $|\tau| < \lambda$. With imaginary values of λ, the function $I_{i\lambda}(\tau)$ tends to infinity for large values of τ and does not oscillate for $\tau > 0$ [13]. The values of λ are real for any k and μ provided that the condition $b^2 < 4$ that matches the Miles condition for the Richardson number is fulfilled. The function $\varphi_1(z) = i(f_1(0)f_2(z) - f_2(0)f_1(z))$ is real and satisfies the boundary condition for $z = 0$. The function $\varphi_2(z) = i(f_1(-\pi)f_2(z) - f_2(-\pi)f_1(z))$ is real and satisfies the boundary condition for $z = -\pi$. Then the characteristic Green function of Eq. (1.6) takes the form

$$\begin{gathered} \varphi(\mu, \nu, z) = -\frac{1}{B(\omega - \mu M(z))^2} \begin{cases} \varphi_1(z)\varphi_2(z_0), & z > z_0, \\ \varphi_1(z_0)\varphi_2(z), & z < z_0, \end{cases} \\ B = \varphi_1(z_0)F_2(z_0) - \varphi_2(z_0)F_1(z_0), \qquad F_j(z) = \frac{\partial \varphi_j(z)}{\partial z}, \quad j = 1, 2 \end{gathered} \tag{2.1}$$

(Wronskian B is independent of z). Introducing $z_- = \min(z, z_0)$ and $z_+ = \max(z, z_0)$, we write the Green function $\varphi(\mu, \nu, z)$ (2.1) as

$$\varphi(\mu, \nu, z) = -\frac{1}{(\omega - \mu M(z))^2}\,\frac{\varphi_1(z_+)\varphi_2(z_-)}{\varphi_1(-\pi)F_2(-\pi)}.$$

Expression (1.5) is integrated with respect to variable μ. The perturbation method can be used to show that the path of integration with respect to μ passes above the real axis on the complex plane μ. The subintegral function amplitude $\varphi(\mu, \nu, z)$ is analytical with respect to μ beyond the poles of this function and the cut L made along the real axis μ from μ_1^* to μ_2^*, where $\mu_1^* = \omega/a$ and $\mu_2^* = \omega/(a - \pi b)$ are the zeroes of the function $I_{\pm i\lambda}$ for $z = 0$ and $z = -\pi$, respectively. In this case, $\mu_1^* = 0.675$ and $\mu_2^* = 3.145$. The point z_0 is critical if $\omega - \mu_0 M(z_0) = 0$, where the corresponding point $\mu_0 \in L$ provided that $z_0 \in [0, -\pi]$. Thus, the critical values on z correspond to the points of the cut L on the complex plane μ. The zeroes of the Wronskian B are the roots of the equation $\varphi_1(-\pi) = 0$. In this case, a dispersion relation can take the form

$$I_{i\lambda}(\beta(\omega - \mu a))I_{-i\lambda}(\beta(\omega - \mu a + \mu b\pi)) = I_{-i\lambda}(\beta(\omega - \mu a))I_{i\lambda}(\beta(\omega - \mu a + \mu b\pi)). \qquad (2.2)$$

The roots of Eq. (2.2) form two series of eigenvalues (dispersed curves) $\mu_{n1}(\nu)$ and $\mu_{n2}(\nu)$. As the values of n become larger, the values of $\mu_{n1}(\nu)$ increase too and tend to μ_1^*, and the values of $\mu_{n2}(\nu)$ drop and tend to μ_2^*. It is noteworthy that, for $\mu = \mu_{nj}(\nu)$ $(j = 1, 2)$, the functions $\varphi_{n1}(\mu, \nu, z)$ and $\varphi_{n2}(\mu, \nu, z)$ are the eigenfunctions of the problem (1.6), which are equal to each other with an accuracy of a constant factor. Therefore, without loss of generality, it can be assumed that $\varphi_{n1}(\mu_{nj}(\nu), \nu, z) = \varphi_{n2}(\mu_{nj}(\nu), \nu, z) = \varphi_{nj}(\nu, z)$ $(j = 1, 2)$. To calculate the integral over the variable μ in Eq. (1.5), it is necessary to close the integration path into the lower half-plane and take into account the integral over the cut L and the sum of the residues in the poles $\mu = \mu_{nj}(\nu)$. We have

$$w(x, y, z) = \sum_{j=1}^{2}\sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty}\int_{0}^{\infty} A_{nj}(\nu, z, z_0)\,\mathrm{e}^{-i(\mu_{nj}(\nu)x + \nu y)}\, d\nu + J,$$

$$A_{nj}(\nu, z, z_0) = \frac{\varphi_{nj}(\nu, z)\varphi_{nj}(\nu, z_0)}{D(\mu_{nj}(\nu), \nu, -\pi)F_{nj}(\nu, -\pi)(\omega - \mu_{nj}(\nu)M(z))^2},$$

$$J = \int_{-\infty}^{\infty} I(\nu, z, z_0)\,\mathrm{e}^{-i\nu y}\, d\nu, \qquad F_{nj}(\nu, z) = \frac{\partial\varphi_{nj}(\nu, z)}{\partial z}, \qquad D(\mu, \nu, z) = \frac{\partial\varphi_1(\mu, \nu, z)}{\partial\mu},$$

where $I(\nu, z, z_0)$ is the integral along the edge of the cut L. It can be shown that the contribution of the integral along the edge of the cut is small as compared to the contribution of the poles $\mu = \mu_{nj}(\nu)$, so the integral J is ignored below. Thus, with account for the harmonic dependence on time, the wave field of the IGWs $W(x, y, z, t)$ can be represented as the sum of the two types of modes

$$W(x, y, z, t) = \sum_{n=1}^{\infty} W_{n1}(x, y, z, t) + W_{n2}(x, y, z, t),$$

$$W_{nj}(x, y, z, t) = \frac{1}{2\pi}\int_{0}^{\infty} A_{nj}(\nu, z, z_0)\,\mathrm{e}^{-i(\mu_{nj}(\nu)x + \nu y - \omega t)}\, d\nu, \quad j = 1, 2. \qquad (2.3)$$

At a distance from the perturbation source and at large values of x, y, the integrals (2.3) in the approximation of the stationary phase method have the form [4, 10]

$$W_{nj}(x, y, z, t) = Z_{nj-} + Z_{nj+},$$

$$Z_{nj\pm} = \frac{A_{nj}(\mu_{nj}(\nu_{nj}^{\pm}), \nu_{nj}^{\pm}, z)}{\sqrt{2\pi x(\pm S_{nj}(\nu_{nj}^{\pm}))}} \cos\left(-i(\mu_{nj}(\nu_{nj}^{\pm})x - \nu_{nj}^{\pm}y \pm \pi/4 + \omega t)\right),$$

$$S_{nj}(\nu) = \frac{d^2\mu_{nj}(\nu)}{d\nu^2}, \qquad j = 1, 2,$$

where $\nu_{nj}^{\pm}$ are the roots of the equation $d\mu_{nj}(\nu)/d\nu = y/x$. Expressions (2.4) are valid only within the corresponding wave wedges, the half-angle θ of each wedge is determined from relations $\theta = \arctan\,(\mu_{nj}(\nu_{nj}^*))$ [ν_{nj}^* is the root of the equation $S_{nj}(\nu_{nj}^*) = 0$]. The asymptotic approximation that describes the wave fields of the IGWs at a distance from the perturbation source and that is valid both near and far from the wave wedges (uniform asymptotics) is expressed via the Airy function and its derivative [4, 10].

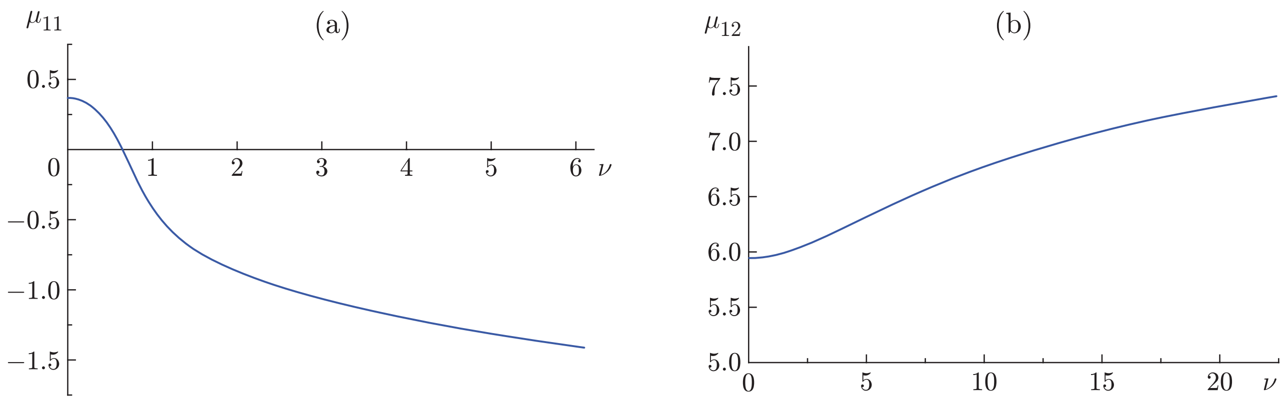

Fig. 1. Dispersion curves of the first mode $\mu_{11}(\nu)$ (a) and $\mu_{12}(\nu)$ (b).

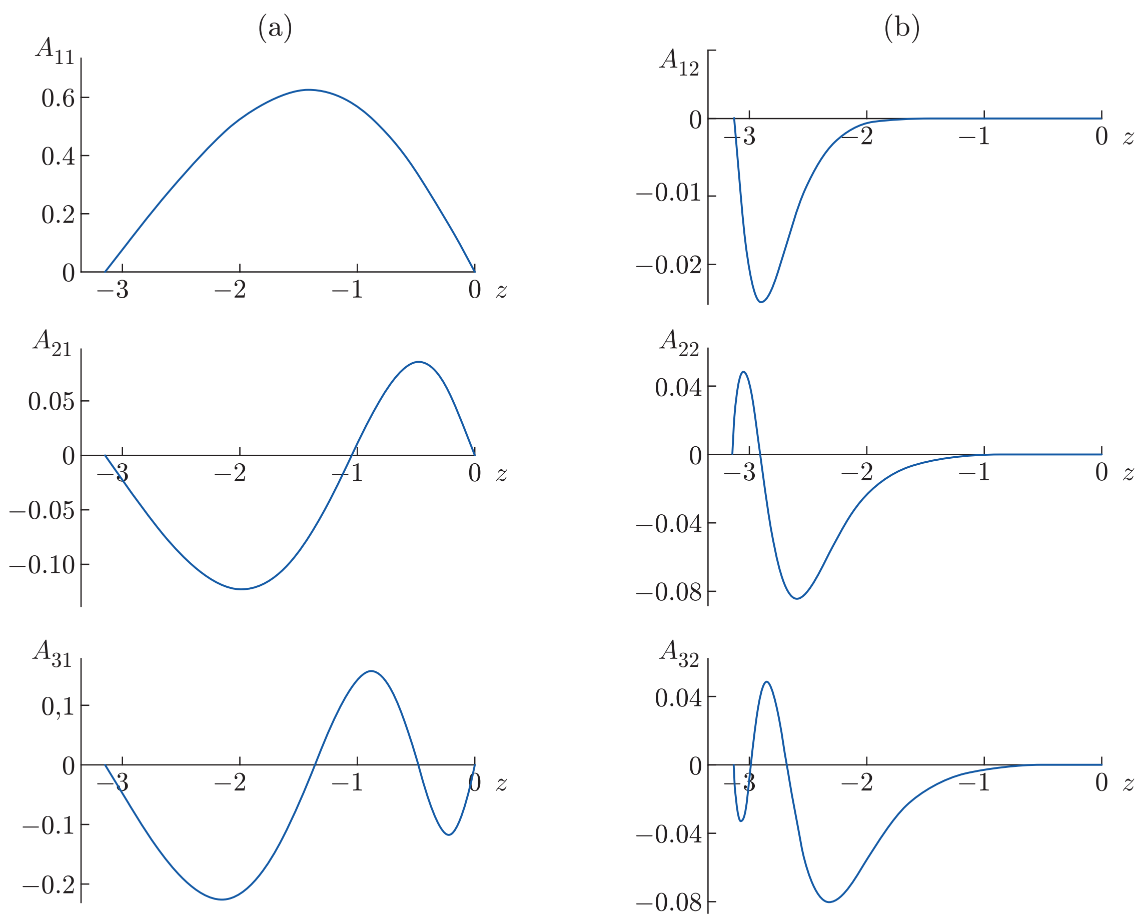

Fig. 2. Subintegral amplitudes for the first three modes of the first type (a) and second type (b).

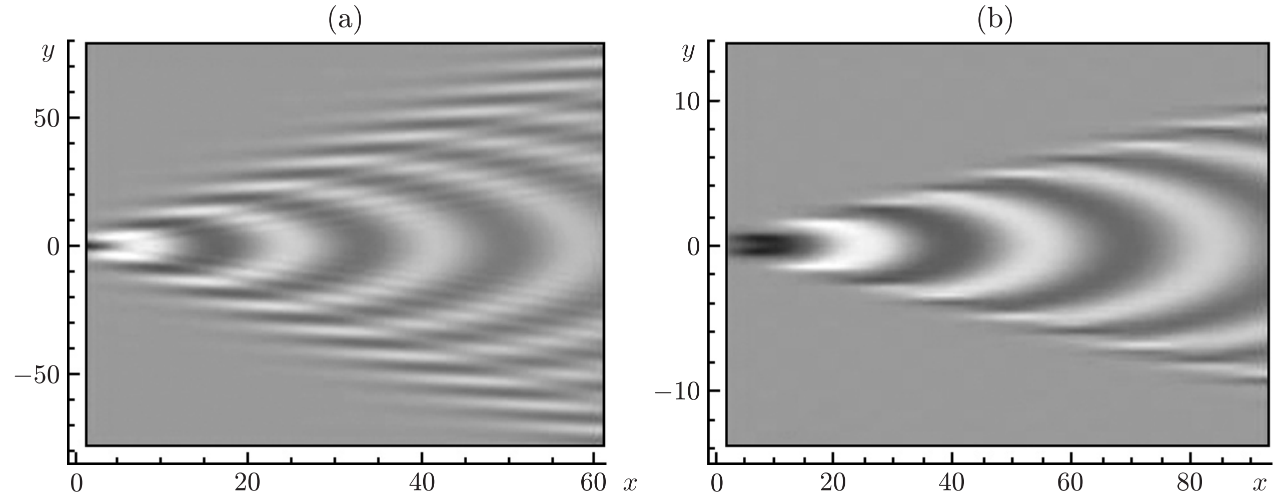

Fig. 3. Wave patterns of the first mode of the first type (a) and second type (b).

3. PHASE AND WAVE PATTERNS

We consider a case with the excited IGW fields. The dispersion curves of the first type intersect the abscissa axis, the half-angle of the wave wedge is less than $\pi/2$, and the wave pattern for the waves of this type is a system of wedge-shaped and transverse wave packets. The corresponding phase structure has the shape of curved triangles nested inside the wave wedges, with their vertex facing the origin. The dispersion curves of the second type are always located above the abscissa axis, in which case the wave pattern is a system of wedge-shaped and longitudinal waves with a simpler phase structure. The wedge half-angle of the waves of the second type is always smaller than that of the waves of the first type. The main contribution to the total IGW field is made by the wave modes of the first type, the amplitudes of the waves of the second type are several times smaller than the amplitudes of the waves of the first type. The half-angle for both types of the waves excited downstream depends on the immersion depth of the perturbation source. In this model of a stratified medium, the shear flow amplitude decreases along with the depth. Therefore, with an increase in the immersion depth of the perturbation source, the ratio of the local-in-depth velocity of the stratified flow to the maximum group velocity of the excited IGWs decreases, which increases the half-angle of the corresponding wave wedge.

Figure 1 shows the dispersion curves of the first mode $\mu_{11}(\nu)$ and $\mu_{12}(\nu)$ for $\nu > 0$ [$\mu_{nj}(\nu)$ denotes the even functions]. Figure 2a shows the first three subintegral amplitudes for the modes of the first type $A_{11}(z)$, $A_{21}(z)$, and $A_{31}(z)$ for $\nu = 0.1$ and $z_0 = -0.9$, and Fig. 2b illustrates the three subintegral amplitudes for the modes of the second type $A_{12}(z)$, $A_{22}(z)$, and $A_{32}(z)$ for $\nu = 0.1$ and $z_0 = -2.5$. Figure 3 shows three-dimensional wave patterns of the first mode of the first type $W_{11}(x, y, z)$ for $z = -0.5$ and $z_0 = -0.9$ and of the second type $W_{12}(x, y, z)$ for $z = -2.9$ and $z_0 = -2.5$.

Thus, the problem of the far field of internal gravity waves excited by the oscillating point source of perturbations in a stratified medium with a shear flow is solved. The analytical solution of the problem is determined using the constant distribution of the buoyancy frequency and the linear dependence of the shear flow velocity on the depth. The solution is represented analytically in the form of the characteristic Green function expressed via the modified Bessel functions of the imaginary index. The expressions for the dispersion relations are obtained and the integral representations of the solutions for the wave fields are constructed. The dependences of the characteristics of the excited far fields of internal gravity waves on the main parameters of the model stratification used, the flows, and the generation regimes are studied.

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