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# ASYMPTOTIC SOLUTIONS OF HIGHER APPROXIMATIONS OF FIELDS OF INTERNAL GRAVITY WAVES IN VARIABLE-DEPTH STRATIFIED MEDIA

**V. V. Bulatov and Yu. V. Vladimirov**

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**Abstract:** A problem of wave dynamics of internal gravity waves in a variable-depth stratified medium is considered. By using a modified method of geometrical optics (vertical modes—horizontal rays), wave modes of higher approximations of asymptotic solutions are constructed. It is demonstrated that the main contributions to the solution in real stratified media are made by the first terms of the corresponding asymptotic presentations.

*Keywords:* internal gravity waves, method of geometrical optics, stratified medium.

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The dynamics of internal gravity waves (IGWs) in natural stratified media (ocean or atmosphere) essentially depends on horizontal inhomogeneity of the medium. The most typical horizontal inhomogeneity in the ocean is the bottom relief. Asymptotic presentations of the solution are usually constructed for an arbitrary shape of the bottom. If the changes in the ocean depth are much smaller than the characteristic IGW length, various generalizations of the space-time ray method can be used for solving problems of wave dynamics. The formulation of sufficient conditions of applicability of this method requires additional considerations.

In finding solutions by the method of geometrical optics or its generalizations, it is usually possible to construct only the first term of the asymptotic presentation because of significant mathematical difficulties, and the higher terms of the asymptotic presentations are assumed to be small. Therefore, construction of asymptotics of higher approximations allows formulation of certain criteria of sufficient conditions for using ray approximations in solving problems of wave dynamics in stratified media with variable parameters. It should be noted that ray approximations are used for the qualitative analysis of the structure of IGW fields in the ocean and clear interpretation of results. Other methods are inapplicable at the moment because of the complexity of problems to be solved.

In this work, we consider a system of equations that describes IGWs excited in a layer of a variable-depth stratified medium by a moving source of perturbations. Within the framework of the linear theory and the Boussinesq approximation, the velocity field in a coordinate system fitted to the moving source satisfies the following system of equations [1, 2]:

$$v^2 \frac{\partial^2}{\partial \xi^{*2}}\Big(\Delta W^* + \frac{\partial^2}{\partial z^{*2}} W^*\Big) + N^2(z^*)\Delta W^* = Qv^2 \delta''_{\xi^*\xi^*}(\xi^*)\, \delta(y^* - y_0^*)\delta'(z^* - z_0^*),$$

$$\Delta U^* + \frac{\partial^2 W^*}{\partial \xi^*\, \partial z^*} = Q\delta'(\xi^*)\delta(y^* - y_0^*)\delta(z^* - z_0^*), \qquad (1)$$

$$\Delta V^* + \frac{\partial^2 W^*}{\partial y^*\, \partial z^*} = Q\delta(\xi^*)\delta'(y^* - y_0^*)\delta(z^* - z_0^*),$$

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$$N^2(z^*) = -\frac{g}{\rho_0(z^*)}\,\frac{d\rho_0(z^*)}{dz^*}, \qquad \Delta = \frac{\partial^2}{\partial \xi^{*2}} + \frac{\partial^2}{\partial y^{*2}}.$$

Here $U$, $V$, and $W$ are the components of velocity of the wave field, $N^2(z^*)$ is the Brunt–Väisälä frequency, and $v$ is the velocity of the source of mass with a power $Q$ ($Q$ is the volume flow rate per second), which uniformly moves at a depth $-z_0^*$ (the $z^*$ axis is directed upward) along the negative direction of the abscissa axis $x^*$ ($\xi^* = x^* + vt$) at a distance $y_0^*$ from the shore. The bottom depth $H^*(y^*)$ is assumed to depend only on the coordinate $y_0^*$, i.e., it does not change along the trajectory of source motion. The boundary conditions are the approximation of a "solid lid" on the surface and the no-slip condition on the bottom:

$$z^* = 0\colon \quad W^* = 0, \qquad z^* = -H^*(y^*)\colon \quad W^* + V^*\,\frac{\partial H^*}{\partial y^*} = 0. \tag{2}$$

Let us consider the case with $N^2(z^*) = \text{const}$. We make the following assumptions about the function $H^*(y^*)$: $H^*(y^*) = 0$, $H^*(0) = 0$ at $y < 0$, $H^*(y^*)$ is a smooth monotonically increasing function satisfying the condition $H^*(y^*) \leqslant \gamma y^*$ at $y^* > 0$, with $\gamma \ll 1$. A liquid layer with a constant depth $H^*(y^*)$ at a fixed value of $y^*$ is called the comparison waveguide (by analogy with underwater acoustic channels in acoustics) [3]. In a stratified layer with a constant depth $H$, the expression for the maximum group velocity of IGWs has the form $C_{\max} = NH/\pi$ [2]. Let us choose the vertical scale as $l = v/N$ and the horizontal scale $L$ as the root of the equation $\pi v = NH^*(L)$. By virtue of conditions imposed on the function $H^*(y^*)$, we have $\lambda = L/l \gg 1$; the dimensionless and dimensional variables are related as follows: $\xi = \xi^*/L$, $y = y^*/L$, $z = z^*/L$, $h(y) = H^*(Ly)/l$, $U = U^* l^2/Q$, $V = V^* l^2/Q$, and $W = W^* l^2/Q$. Then, the dimensionless value of $Y$ determined from the equation $h(y^*) = \pi$ corresponds to the comparison waveguide depth $h(y^*)$, for which we have $C_{\max}(Y) = v$.

We also assume that the source motion velocity $v$ is greater than $C_{\max}(y_0)$, i.e., $y_0 < Y$ [by virtue of monotonicity of $H(y^*)$]. Thus, the maximum group velocity of IGWs for the corresponding comparison waveguides is smaller than the source velocity in the domain $y < Y$ and greater than the source velocity in the domain $y > Y$.

In dimensionless variables, system (1) and the boundary conditions (2) have the form

$$\begin{gathered}\frac{\partial^2}{\partial \xi^2}\Big(\frac{1}{\lambda^2}\,\Delta + \frac{\partial^2}{\partial z^2}\Big)W + \Delta W = \frac{1}{\lambda^2}\,\delta''_{\xi\xi}(\xi)\delta(y-y_0)\delta'(z-z_0),\\ \frac{1}{\lambda}\,\Delta V + \frac{\partial^2 W}{\partial y\,\partial z} = \frac{1}{\lambda^2}\,\delta(\xi)\delta'_y(y-y_0)\delta(z-z_0);\end{gathered} \tag{3}$$

$$z = 0\colon \quad W = 0, \qquad z = -h(y)\colon \quad W + h'(y)V/\lambda = 0. \tag{4}$$

The solution of problem (3), (4) is sought in the form typical for the method of geometrical optics ($\omega = \omega^*/N$ is the dimensionless frequency normalized to $N$) [1, 4–6]

$$\begin{gathered}W = \int F(z,y,\omega)\exp\big(i\lambda(\omega\xi - S(y,\omega))\big)\,d\omega,\\ V = \int \Psi(z,y,\omega)\exp\big(i\lambda(\omega\xi - S(y,\omega))\big)\,d\omega,\end{gathered} \tag{5}$$

where the amplitude factors $F$ and $\Psi$ are presented in the form of series in inverse powers of the parameter $\lambda$ [1, 4–6]:

$$F(z,y,\omega) = \sum_{m=0}^{\infty}\Big(\frac{i}{\lambda}\Big)^m F_m(z,y,\omega), \qquad \Psi(z,y,\omega) = i\sum_{m=0}^{\infty}\Big(\frac{i}{\lambda}\Big)^m \Psi_m(z,y,\omega). \tag{5$'$}$$

Substituting the solutions for $W$ and $V$ from Eq. (5) into Eqs. (3), taking into account Eqs. (5$'$), and equating terms at identical powers of $\lambda$, we obtain

$$\begin{gathered}F''_{0\,zz} + \frac{\varkappa^2}{c^2}\,F_0 = 0, \qquad F''_{1\,zz} + \frac{\varkappa^2}{c^2}\,F_1 = -\frac{1}{c^2}\,(2S'F'_0 + S''F_0),\\ F''_{m\,zz} + \frac{\varkappa^2}{c^2}\,F_m = -\frac{1}{c^2}\,(2S'F'_{m-1} + S''F_{m-1} + F''_{m-2}),\end{gathered} \tag{6}$$

where $m = 2, 3, \ldots$, $\varkappa^2 = \omega^2 + S'_y(y,\omega)$, and $c^2 = \omega^2/(1-\omega^2)$; the primes in the right sides denote the derivatives with respect to the horizontal variable $y$.

The boundary conditions have the following form:
—on the surface,

$$z = 0: \qquad F_m = 0, \quad m = 0, 1, 2, 3, \dots;$$

—on the bottom,

$$z = -h(y): \qquad F_0 = 0, \quad F_m + \Psi_{m-1} h'(y) = 0, \quad m = 1, 2, \dots . \tag{7}$$

The functions $\Psi_m(z, y, \omega)$ are determined from the following chain of equalities (obtained by substituting Eqs. (5) and (5′) into the second equation of system (3) and equating terms with identical powers of $\lambda$):

$$\begin{gathered} \Psi_0 = -\frac{S' F'_{0\,z}}{\varkappa^2}, \qquad \Psi_1 = -\frac{S' F'_{1\,z} + 2\Psi'_0 S' + \Psi_0 S'' + F''_{0\,yz}}{\varkappa^2}, \\ \Psi_m = -\frac{S' F'_{m\,z} + 2\Psi'_{m-1} S' + \Psi_{m-1} S'' + F''_{m-1\,yz} + \Psi''_{m-2}}{\varkappa^2}. \end{gathered} \tag{8}$$

## ZEROTH APPROXIMATION (DETERMINATION OF THE FUNCTIONS $F_0$ AND $\Psi_0$)

Let us consider the first equation of system (6):

$$\begin{gathered} F_{0\,zz} + \frac{\varkappa^2}{c^2 F_0} = 0, \\ z = 0,\; z = -h(y): \qquad F_0 = 0. \end{gathered} \tag{9}$$

The boundary-value problem (9) yields a set of eigenvalues (local dispersion relations)

$$\varkappa_n(y, \omega) = \frac{\omega n \pi}{\sqrt{1 - \omega^2}\, h(y)}, \qquad n = 1, 2, \dots$$

and eigenfunctions

$$F_{0n} = A_{0n}(y, \omega) \sin \frac{n\pi z}{h(y)}, \qquad n = 1, 2, \dots \tag{10}$$

with an indefinite amplitude dependence $A_{0n}(y, \omega)$. Then, the expression for the eikonal $S(y, \omega)$ is

$$S_n(y, \omega) = \omega \sqrt{\frac{n^2 \pi^2}{(1 - \omega^2) h^2(y)} - 1}, \qquad n = 1, 2, \dots .$$

To find the function $A_{0n}(y, \omega)$ independent of $z$, we use the condition of solvability of the boundary-value problem for $F_1$ [second equation in (6)], which is obtained by multiplying the equation for $F_1$ by $F_{0n}$ and integrating it with respect to $z$ from $-h(y)$ to 0. As a result, we obtain

$$F'_{0n\,z} F_{1n}\Big|_{z=-h(y)} = -\frac{1}{c^2} \int\limits_{-h(y)}^{0} \frac{\partial}{\partial y} (F_{0n}^2 S')\, dz.$$

Taking into account the boundary conditions on the bottom (7), (8), we obtain the expression for $F_{1n}$:

$$F_{1n}\Big|_{z=-h(y)} = \frac{S'_n h' F'_{0n\,z}}{\varkappa_n^2}\Big|_{z=-h(y)}.$$

Taking into account Eq. (10) for $F_{0n}$, we obtain the transport equation for $A_{0n}(y, \omega)$:

$$S'_n(y, \omega) h'(y) A_{0n}^2(y, \omega) = -\frac{1}{2} \frac{\partial}{\partial y} (S'_n(y, \omega) h(y) A_{0n}^2(y, \omega)). \tag{11}$$

The transport equation (11) yields the conservation law

$$h^3(y) S'_n(y, \omega) A_{0n}^2(y, \omega) = \text{const},$$

from which we find

$$A_{0n}^2(y, \omega) = \frac{h(y_0)}{h(y)} \sqrt[4]{\frac{n^2\pi^2 - (1 - \omega^2) h^2(y_0)}{n^2\pi^2 - (1 - \omega^2) h^2(y)}}\, A_{0n}(y_0, \omega). \tag{12}$$

The function $A_{0n}(y_0, \omega)$ is determined on the basis of the locality principle [1, 4], i.e., from the solution of the problem for a layer of a constant depth $h(y_0)$:

$$A_{0n}(y_0, \omega) = \frac{2ic \sin(n\pi z_0 / h(y_0))}{\pi \sqrt{n^2\pi^2 - (1 - \omega^2) h^2(y_0)}}.$$

Thus, the function $F_{0n}(z, y, \omega)$ is determined. The function $\Psi_{0n}$ is determined from the first equation of (8).

## CALCULATION OF THE AMPLITUDES OF HIGHER APPROXIMATIONS ($m \geqslant 1$)

First we consider the case with $m = 1$ (at $m > 1$, the algorithm of finding the amplitudes $A_{mn}$ is not principally different, but the calculation procedure is more complicated). The procedure of finding the amplitudes $A_{mn}$ is similar to the procedure of determining the amplitudes of acoustic waves [3, 4].

The function $F_1(z, y, \omega)$ is sought by solving the boundary-value problem

$$F''_{1n\,zz} + \frac{n^2\pi^2}{h^2(y)}\,F_{1n} = -\frac{1}{c^2}\,(2S'_n F'_{0n} + S''_n F_{0n}),$$
$$z = 0\colon\quad F_{1n} = 0, \qquad z = -h(y)\colon\quad F_{1n} = \frac{S'_n(y,\omega)h'(y)F'_{0n\,z}}{\varkappa_n^2(y,\omega)}. \tag{13}$$

The solution of problem (13) is sought in the form

$$F_{1n}(z, y, \omega) = a_1(y, \omega)\sin\frac{n\pi z}{h(y)} + B_{1n}(z, y, \omega). \tag{14}$$

Here $B_{1n}(z, y, \omega)$ is the partial solution of Eq. (13) satisfying the boundary condition at $z = 0$; in this case, the boundary condition on the bottom is satisfied automatically in view of the specially determined function $A_{0n}(y, \omega)$.

The function $B_{1n}(z, y, \omega)$ can be found, for instance, by using the method of variation of the constants:

$$B_{1n}(z, y, \omega) = \frac{S'_n(y,\omega)A_{0n}(y,\omega)}{2n\pi c^2 h(y)}\Big(n\pi z^2\sin\frac{n\pi z}{h(y)} - 2z\cos\Big(\frac{n\pi z}{h(y)}\Big)h(y)\Big)h'(y).$$

From Eq. (14), we find the amplitude of the eigenfunction $a_1(y, \omega)$, which is determined from the condition of orthogonality of the right side of the boundary-value problem for $F_{2n}(z, y, \omega)$:

$$F''_{2n\,zz} + \frac{n^2\pi^2}{h^2(y)}\,F_{2n} = -\frac{1}{c^2}\,(2F'_{1n}S'_n + F_{1n}S''_n + F''_{0n}),$$
$$z = 0\colon\quad F_{2n} = 0, \qquad z = -h(y)\colon\quad F_{2n} = h'(y)\,\frac{2\Psi'_{0n}S'_n + \Psi_{0n}S''_n + F''_{0n\,yz} + F'_{1n\,z}S'_n}{\varkappa^2} \tag{15}$$

and the amplitude of the eigenfunction $\sin(n\pi z/h(y))$.

Multiplying Eqs. (15) by $\sin(n\pi z/h(y))$ and integrating them with respect to $z$ from 0 to $-1$, we obtain

$$\frac{n\pi h'(y)\cos n\pi}{h(y)}\;\frac{S'_nF'_{1n\,z} + 2\Psi'_{0n}S'_n + \Psi_{0n}S''_n + F''_{0n\,yz}}{\varkappa_n^2}\Big|_{z=-h(y)} = -\frac{1}{c^2}\int\limits_{-h(y)}^{0}(2F'_{1n}S'_n + F_{1n}S''_n + F''_{0n})\sin\frac{n\pi z}{h(y)}\,dz, \tag{16}$$

where the function $F_{1n}$ is determined by Eq. (14).

With respect to the amplitude $a_{1n}(y, \omega)$, Eq. (16) is a linear inhomogeneous first-order differential equation, which can be presented as

$$\frac{\partial a_{1n}(y,\omega)}{\partial y} + \frac{\partial \ln r(y,\omega)}{\partial y}\,a_1(y,\omega) = R(y,\omega), \tag{17}$$

where $r(y, \omega) = \ln[h(y)(n^2\pi^2 - (1-\omega^2)h^2(y))^{1/4}]$; the right side $R(y, \omega)$ is a known function.

In accordance with the locality principle, the initial condition has the form

$$a_{1n}(y_0, \omega) = 0.$$

The solution of Eq. (17) satisfying this initial condition is written as

$$a_{1n}(y,\omega) = \frac{1}{r(y,\omega)}\int\limits_{y_0}^{y} r(\eta,\omega)R(\eta,\omega)\,d\eta. \tag{18}$$

Equation (18) is a typical equation for asymptotic solutions of higher approximations obtained by the method of geometrical optics and its modifications [3, 4]. Note that the transport equation (17) with a zero right side yields the same conservation law (12) as the homogeneous transport equation (11). Thus, the function $F_{1n}(z, y, \omega)$ is completely determined.

The function $\Psi_1(z, y, \omega)$ is determined from the second equation of (8). Higher approximations of $F_m$ and $\Psi_m$ at $m > 1$ can be constructed in a similar manner.

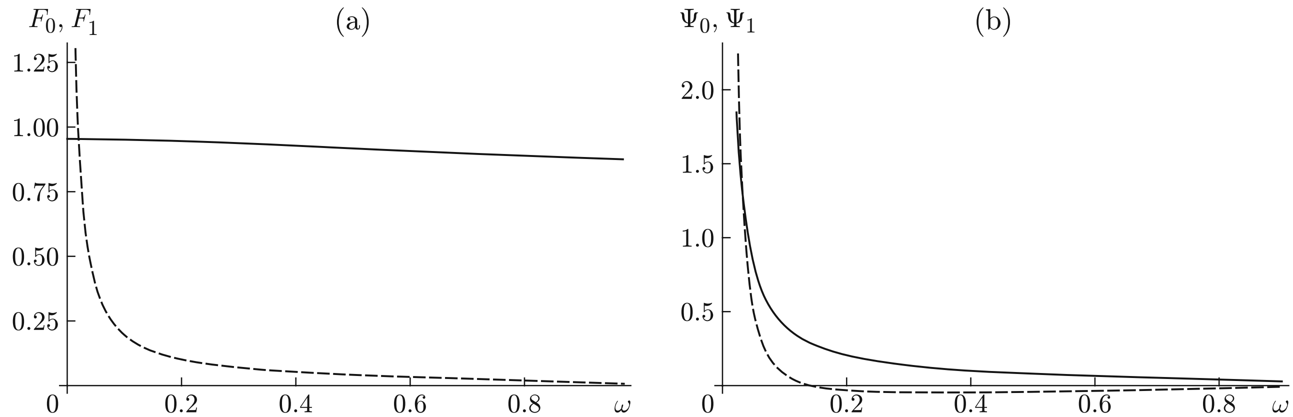


**Fig. 1.** The first approximation $F_0$ (solid curves) and second approximation $F_1$ (dashed curves) of the vertical IGW velocity (a) and horizontal IGW velocity (b) versus the frequency $\omega$ at $z = -1.2$ and $y = 0.8$.

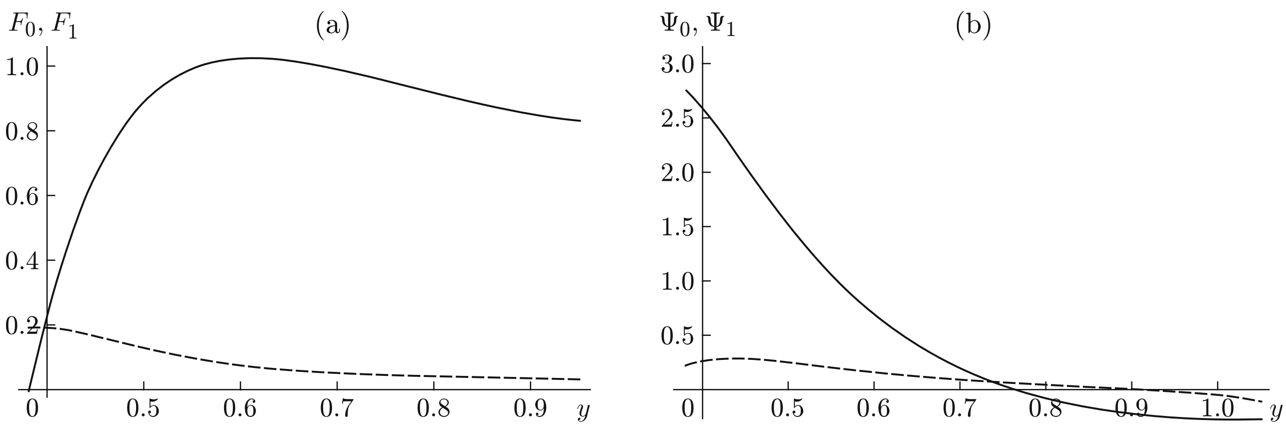


**Fig. 2.** The first approximation $F_0$ (solid curves) and second approximation $F_1$ (dashed curves) of the vertical IGW velocity (a) and horizontal IGW velocity (b) versus the horizontal coordinate $y$ at $z = -1.2$ and $\omega = 0.5$.

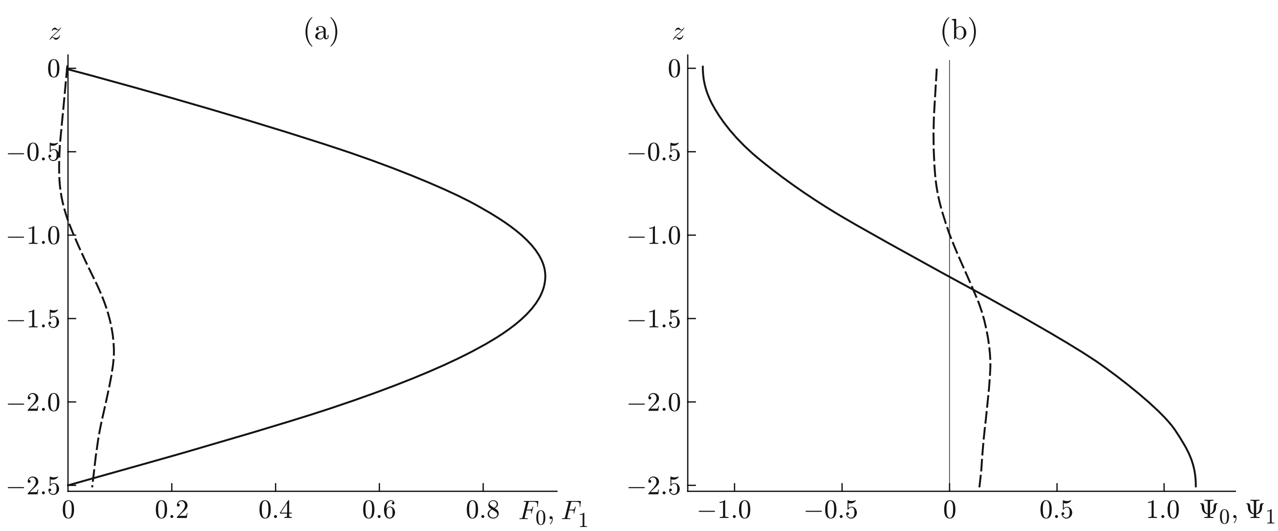


**Fig. 3.** The first approximation $F_0$ (solid curves) and second approximation $F_1$ (dashed curves) of the vertical IGW velocity (a) and horizontal IGW velocity (b) versus the vertical coordinate $z$ at $y = 0.8$ and $\omega = 0.5$.

## RESULTS OF NUMERICAL CALCULATIONS

The numerical calculations were performed to compare the functions $F_0(z, y, \omega)$ with $F_1(z, y, \omega)$ and $\Psi_0(z, y, \omega)$ with $\Psi_1(z, y, \omega)$. The calculations were performed for the case of a linear profile of the bottom with an angle of 7° and the first mode ($n = 1$) typical for the Arctic basin region [7, 8]. The slope of 7° corresponds to the parameters $\lambda = 44.8$ and $y_0 = 0.7$. All calculations were performed in the "subcritical domain" $y_0 < y < y_*$, where $y_*$ is the turning point, i.e., $y_*$ is the root of the equation $S'_{1y}(y, \omega) = 0$; in this case, we have $y_* = 1/\sqrt{1 - \omega^2}$.

Figures 1–3 show the dependences of $F_0$ and $\Psi_0$ (solid curves) and $F_1$ and $\Psi_1$ (dashed curves) on the frequency $\omega$, horizontal coordinate $y$, and vertical coordinate $z$.

The analytical and numerical results obtained show that higher approximations of the ray method in real stratified media are small as compared with the first term of the asymptotic solution. Thus, for the qualitative analysis of the structure of IGW fields and interpretation of full-scale measurements in the ocean, it is possible to use only the first term of the ray approximations of the solutions.

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