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Asymptotical Analysis of Internal Gravity Wave Dynamics in Stratified Medium

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Abstract

In paper fundamental problems of internal gravity waves dynamics are considered. Wave dynamics of stratified medium (ocean, atmosphere) is highly dependent on density distribution and bottom topography. The exact analytical solution is obtained only if the water distribution density and bottom shape described by some model functions. The solution of this problem is expressed in terms of the Green's function and the asymptotic representations of the solutions are considered. The uniform asymptotic forms of the internal gravity waves in horizontally inhomogeneous and non-stationary stratified ocean are obtained. A modified spatio-temporal ray method is proposed, which belongs to the class of geometrical optics methods (WKBJ method). The solution is proposed in terms of wave modes, propagating independently at the adiabatic approximation, and described as a non-integral degree series of a small parameter characterizing the stratified medium properties. Analytical and numerical algorithms of internal gravity wave calculations for the real ocean parameters are presented.

Keywords: WKBJ approximations, wave propagation, stratified medium, internal gravity waves

Introduction

Industrial activities on the continental shelf connected with oil, gas, and other minerals extraction became one of the important reasons to begin researches of dynamic internal gravity waves [1-3]. Ships and platforms busy with drilling and construction at the depth use long tubes joining them with the sea bottom. Builders of underwater constructions in equatorial districts experienced the influence of huge underwater internal waves and strong surface flows which can have the form of steep waterfalls. Some time ago when the phenomenon of internal waves and their strength were not known it happened that the builders lost their equipment. Such expensive losses made them think that security of underwater equipment and the influence of internal gravity waves should be controlled. The internal waves characteristics are used for appreciation of their influence on the environment and underwater platforms of oil and gas deposits at the shelf (Arctic basin, China and Yellow Seas, etc) [4-12]. Special interest to research of internal gravity waves is connected with also intensive exploitation of Arctic and its natural wealth. Internal gravity waves in Arctic are poorly studied as they move under ice and practically invisible from above, but accessible information about underwater objects movement show their existence. Sometimes there are exclusions when internal gravity waves reach ice and uplift and lower it with definite periodicity which can be fixed with the help of radiolocation sounding. Influence of all kinds of waves can be the reason of the ice cover split in the Arctic. Internal waves make for the movement of icebergs and different kinds of pollution. So, the research of internal gravity waves dynamics is an important fundamental scientific and practical problem aimed at ensuring security while [8,9].

The internal gravity waves are the oscillations of a stratified medium in the gravity force field. The stratified medium is such a medium where the density increases with the depth. Suppose that a volume element of the medium is not at the equilibrium, for example it could be displaced upward, then it will be heavier than the surrounding medium and therefore Archimedean forces will make it move back to the equilibrium. The essential parameter of any oscillating system is the frequency. It is determined by the correlation of two factors: returning forces which return the perturbed system towards its equilibrium and the inertial forces. For the internal gravity waves the returning forces are proportional to the vertical gradient of the fluid's density and the inertial ones are proportional to the density itself. For the characteristic frequency of the gravity waves oscillations we have the following expression: $N^2(z) = -\frac{g}{\rho(z)}\frac{d\rho(z)}{dz}$. This frequency is usually called by the Brunt-Vaisala frequency or the buoyancy frequency. Here $\rho(z)$ is the density considered as a function of the depth z , g is the acceleration in the gravity force field, the sign “-” originates from the increase of the density with the depth and therefore $\frac{d\rho(z)}{dz} < 0$ [1-3].

The exact solutions of the essential equations describing the internal gravity waves are only obtained for special cases. That is the reason why the approximate asymptotical methods are systematically used for the investigation of the internal gravity wave fields in stratified ocean. The internal gravity waves are usually represented in the following integral form:
$J = \int_{\gamma} \exp[\lambda f(z)] F(z)\, dz, \quad \lambda >> 1$, where $f(z)$ and $F(z)$ are analytic functions of the complex variable z; γ is a contour of integration on the complex plane z. The universal way to construct the asymptotic forms of such expressions is the method of etalon integrals [13-17].

This paper is devoted to the systematical description of a generalization of the geometrical optics method (WKBJ method), i.e. we discuss the spatio-temporal ray method of etalon functions . This method allows one to solve the problem of asymptotic modeling of the inharmonic wave packet’s dynamics for the internal gravity waves in stratified media with slowly varying parameters. The main reasons to use the ray methods are the following: the ray representations are well correlated with the intuition and with the empirical material for the propagation of the internal gravity waves in natural stratified media (ocean, atmosphere). These methods are universal and very often one can use only them for the approximate computations of the wave fields in slowly changing non-homogeneous stratified media [2,13,16,17].

As is well known, an essential influence on the propagation of internal gravity waves in stratified natural media (ocean, atmosphere) is caused by the horizontal inhomogeneity and non-stationarity of these media. To the most typical horizontal inhomogeneity of a real ocean one can refer the modification of the relief of the bottom, an inhomogeneity of the density field, and the variability of the mean flows. One can obtain an exact analytic solution of this problem (for instance, by using the method of separation of variables) only if the distribution of density and the shape of the bottom are described by rather simple model functions. If the shape of the bottom and the stratification are arbitrary, then one can construct only asymptotic representations of the solution in the near and far zones; however, to describe the field of internal waves between these zones, one needs an accurate numerical solution of the problem [1-3].

Using asymptotic methods, one can consider a wide class of interesting physical problems, including problems concerning the propagation of non-harmonic wave packets of internal gravity waves in diverse nonhomogeneous stratified media under the assumption that the modification of the parameters of a vertically stratified medium are slow in the horizontal direction. From the general point of view, problems of this kind can be studied in the framework of a combination of the adiabatic and semi-classical approximations or by using close approach, for example, ray expansions. In particular, the asymptotic solutions of diverse dynamical problems can be described by using the Maslov canonical operator, which determines the asymptotic behavior of the solutions, including the case of neighborhoods of singular sets composed of focal points, caustics, etc [13,14,17]. The specific form of the wave packet can be finally expressed by using some special functions, say, in terms of oscillating exponentials, Airy function, Fresnel integral, Pearcey-type integrals, etc. The above approaches are quite general and, in principle, enable one to solve a broad spectrum of problems from the mathematical point of view; however, the problem of their practical applications and, in particular, of the visualization of the corresponding asymptotic formulas based on the Maslov canonical operator is still far from completion, and in some specific problems one can use other schemes to find the asymptotic behavior whose computer realization using software of *Mathematica* type is rather simple. In this paper, using the approaches developed in [1,15], we construct asymptotic solutions of the problem which is formulated as follows.

Problem formulation: horizontally inhomogeneous vertically stratified medium

This paper covers the conceptual issues of the spatio-temporal method (the geometrical optics method , the WKBJ approximation) taking into consideration the specifics of internal gravity waves. If we examine the internal gravity waves for the case when the undisturbed density field $\rho_0(z,x,y)$ depends not only on the depth z , but on the horizontal coordinates x and y , then, in general terms, if the undisturbed density is a function of horizontal coordinates, such a distribution of density induces a field of horizontal flows. These flows, however, are extremely slow and in the first approximation can be neglected. So it is commonly supposed that the field $\rho_0(z,x,y)$ is defined a priori, thus, it is assumed that there exist certain external sources or the examined system is non-conservative. It is also evident that if the internal gravity waves are propagating above an non-uniform bottom there is no such a problem, because the "internal wave – non-uniform bottom" system is conservative and there is no external energy flush [1-3].

Next we shall write out the linearized system of equations of hydrodynamics [1-3,15]

$$\rho_0\frac{\partial\tilde{U}_1}{\partial t}=-\frac{\partial p}{\partial x}, \qquad \rho_0\frac{\partial\tilde{U}_2}{\partial t}=-\frac{\partial p}{\partial y},$$

$$\rho_0\frac{\partial\tilde{W}}{\partial t}=-\frac{\partial p}{\partial z}+g\rho, \tag{1}$$

$$\frac{\partial\tilde{U}_1}{\partial x}+\frac{\partial\tilde{U}_2}{\partial y}+\frac{\partial\tilde{W}}{\partial z}=0,$$

$$\frac{\partial\rho}{\partial t}+\tilde{U}_1\frac{\partial\rho_0}{\partial x}+\tilde{U}_2\frac{\partial\rho_0}{\partial y}+\tilde{W}\frac{\partial\rho_0}{\partial z}=0.$$

Here $(\tilde{U}_1, \tilde{U}_2, \tilde{W})$ is the velocity vector of internal gravity waves, p and ρ are the pressure and density perturbations, g is the acceleration of gravity (z-axis is directed downwards). Using the Boussinesq approximation which means the density $\rho_0(z,x,y)$ in the first three equations of the system (1) is assumed a constant value, the system (1) by applying the cross-differentiating will be given as

$$\frac{\partial^4 \tilde{W}}{\partial z^2 \partial t^2} + \Delta \frac{\partial^2 \tilde{W}}{\partial t^2} + \frac{g}{\rho_0} \Delta (\tilde{U}_1 \frac{\partial \rho_0}{\partial x} + \tilde{U}_2 \frac{\partial \rho_0}{\partial y} + \tilde{W} \frac{\partial \rho_0}{\partial z}) = 0 ,$$

$$\frac{\partial}{\partial t} (\Delta \tilde{U}_1 + \frac{\partial^2 \tilde{W}}{\partial z \partial x}) = 0 , \qquad (2)$$

$$\frac{\partial}{\partial t} (\Delta \tilde{U}_2 + \frac{\partial^2 \tilde{W}}{\partial z \partial y}) = 0 ,$$

$$\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 .$$

As the boundary conditions we take the "rigid-lid" condition

$$W = 0 \quad \text{at} \quad z = 0, H . \qquad (3)$$

Consider the harmonic waves $(\tilde{U}_1, \tilde{U}_2, \tilde{W}) = \exp(i\omega t)(U_1, U_2, W)$. Introduce the non-dimensional variable according to the formulas: $x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{h},$ where L is the typical scale of the horizontal variations ρ_0, h is the typical scale of the vertical variations ρ_0 (for example, the thermocline width) [13,15].

In non-dimension coordinates the equation system (2) will be written as (index $*$ is omitted hereafter)

$$-\omega^2 (\frac{\partial^2 W}{\partial z^2} + \varepsilon^2 \Delta W) + \varepsilon^2 \frac{g_1}{\rho_0} (\varepsilon U_1 \frac{\partial \rho_0}{\partial x} + \varepsilon U_2 \frac{\partial \rho_0}{\partial y} + W \frac{\partial \rho_0}{\partial z}) = 0,$$

$$\varepsilon \Delta U_1 + \frac{\partial^2 W}{\partial z \partial x} = 0 , \qquad \varepsilon \Delta U_2 + \frac{\partial^2 W}{\partial z \partial y} = 0 , \qquad (4)$$

where $\varepsilon = \frac{h}{L} << 1, \qquad g_1 = \frac{g}{h}$.

Asymptotic solutions

The asymptotic solution (4) shall be found in the form usual for the geometric optics method [13,17,18-20]

$$\mathbf{V}(z,x,y) = \sum_{m=0}^{\infty} (i\varepsilon)^m \mathbf{V}_m (z,x,y) \exp(\frac{S(x,y)}{i\varepsilon}) , \qquad (5)$$

$$\mathbf{V}(z,x,y) = (U_1(z,x,y), U_2(z,x,y), W(z,x,y)) .$$

Functions $S(x,y)$ and $\mathbf{V}_m, m=0,1,...$ are subject to definition. From here on we shall restrict ourselves to finding only the dominant member of the expansion (5) for the vertical velocity component $W_0(z,x,y)$, at that from the last two equations (4) we have

$$U_{10} = -\frac{i\partial S/\partial x}{|\nabla S|^2}\frac{\partial W_0}{\partial z}, \quad (6)$$

$$U_{20} = -\frac{i\partial S/\partial y}{|\nabla S|^2}\frac{\partial W_0}{\partial z},$$

$$|\nabla S| = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2.$$

Substitute (5) into the first equation of the system (4) and set equal the members of the order $O(1)$

$$\frac{\partial^2 W_0}{\partial z^2} + |\nabla S|^2\left(\frac{N^2(z,x,y)}{\omega^2} - 1\right)W_0 = 0, \quad (7)$$

$$W_0(0,x,y) = W_0(H,x,y) = 0,$$

where $N^2(z,x,y) = \frac{g_1}{\rho_0}\frac{\partial\rho_0}{\partial z}$ is the Vaisala-Brunt frequency depending of the horizontal coordinates. The Vaisala-Brunt frequency (buoyancy frequency) is a main parameter than determines internal gravity wave properties in real ocean [1-3]. The boundary problem (7) has a calculation setup of eigenfunctions W_{0n} and eigenvalues $K_n(x,y) \equiv |\nabla S_n|$, which are assumed to be known [15] . From here on the index n will be omitted while assuming that further calculations belong to an individually taken mode.

For the function $S(x,y)$ we have the eikonal equation [14,18-20]

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = K^2(x,y) \quad (8)$$

Initial conditions for the eikonal S for the horizontal case are defined on the line $L: x_0(\alpha), y_0(\alpha)$: $S(x,y)\big|_L = S_0(\alpha)$ [18-20]. For solving the eikonal equation we construct the rays, that is, the equation (8) with characteristics

$$\frac{dx}{d\sigma} = \frac{p}{K(x,y)}, \qquad \frac{dp}{d\sigma} = \frac{\partial K(x,y)}{\partial x}, \quad (9)$$

$$\frac{dy}{d\sigma} = \frac{q}{K(x,y)}, \qquad \frac{dq}{d\sigma} = \frac{\partial K(x,y)}{\partial y},$$

where $p = \partial S/\partial x$, $q = \partial S/\partial y$, $d\sigma$ is the length element of the ray.

The initial conditions p_0 and q_0 shall be defined from the system

$$p_0\frac{\partial x_0}{\partial\alpha} + q_0\frac{\partial y_0}{\partial\alpha} = \frac{\partial S_0}{\partial\alpha},$$

$$p_0^2 + q_0^2 = K^2(x_0(\alpha), y_0(\alpha)).$$

The equations (9) and initial conditions $x_0(\alpha), y_0(\alpha), p_0(\alpha), q_0(\alpha)$ define the ray $x = x(\sigma,\alpha),\ y = y(\sigma,\alpha)$. After the rays are found the eikonal S is defined by integration along the ray [15,16,19]

$$S = S_0(\alpha) + \int_0^\sigma K(x(\sigma,\alpha), y(\sigma,\alpha))d\sigma . \qquad (10)$$

Now we are going to find the eigenfunction $W_0(z,x,y)$. Note that from (7) we can determine only the vertical dependency of the function $W_0(z,x,y)$. In other words, the function W_0 is defined to the accuracy of multiplication by the arbitrary function $A_0(x,y)$. We shall find W_0 given as

$$W_0(z,x,y) = A_0(x,y)\hat{W}_0(z,x,y), \qquad (11)$$

where $\hat{W}_0(z,x,y)$ is the solution of the problem (7) normalized as follows

$$\int_0^H (N^2(z,x,y) - \omega^2)\hat{W}_0^2(z,x,y)dz = 1. \qquad (12)$$

After substituting (5) into (4) set equal the members of the order $O(\varepsilon)$

$$\omega^2\left(\frac{\partial^2 W_1}{\partial z^2} + K^2\left(\frac{N^2(z,x,y)}{\omega^2} - 1\right)W_1\right) =$$

$$= (\omega^2 - N^2)(2\nabla W_0 \nabla S + W_0 \Delta S) + \frac{g_1}{\rho_0}(\nabla S \nabla \rho_0)\frac{\partial W_0}{\partial z} - 2(\nabla N^2 \nabla S)W_0, \qquad (13)$$

$$W_1(0,x,y) = W_1(H,x,y).$$

Next we use the orthogonality condition of the right-hand member of the equation (13) with respect to function $W_0(z,x,y)$. Multiplying (13) by W_0 and integrating over z from 0 to H, we obtain

$$\int_0^H (N^2(z,x,y) - \omega^2)\nabla(W_0^2 \nabla S)dz - \frac{g_1}{2}\int_0^H (\nabla S \nabla \ln \rho_0)(\frac{\partial W_0^2}{\partial z})dz + 2\int_0^H (\nabla N^2(z,x,y)\nabla S)W_0^2 dz = 0$$

. (14)

Convert the second term into (14) using the integration by parts and zero boundary conditions for W_0

$$-\frac{g_1}{2}\int_0^H (\nabla S \nabla \ln \rho_0)(\frac{\partial W_0^2}{\partial z})dz = \frac{\nabla S}{2}\int_0^H \nabla N^2 W_0^2 dz . \qquad (15)$$

Convert the first term into (14) while accounting for (12)

$$\int_0^H (N^2 - \omega^2)\nabla(W_0^2 \nabla S)dz = \nabla(A_0^2 \nabla S) - \int_0^H (\nabla S \nabla N^2)W_0^2 dz . \qquad (16)$$

In order to transform the third member into (14) we apply the gradient operator to the equation (7) setting $\mathbf{Y} = \nabla W_0$

$$\frac{\partial^2 \mathbf{Y}(z,x,y)}{\partial z^2} + K^2(x,y)\left(\frac{N^2(z,x,y)}{\omega^2} - 1\right)\mathbf{Y} +$$

$$+W_0 \nabla\left[K^2(x,y)\left(\frac{N^2(z,x,y)}{\omega^2} - 1\right)\right] = 0 \tag{17}$$

Multiplying (17) by W_0 and integrating over z from 0 to H while accounting for (12) we get

$$\int_0^H W_0^2 \nabla N^2(z,x,y)dz = -2A_0^2(x,y)\nabla \ln K(x,y). \tag{18}$$

Finally, we write over the equation (14) using (15), (16) and (18)

$$\nabla A_0^2 \nabla S + A_0^2 \Delta S - 3\nabla S \nabla \ln K = 0. \tag{19}$$

The transfer equation (19) will be solved in characteristics of the eikonal equation (9). Using the formula for ΔS along the rays: $\Delta S = \frac{1}{J}\frac{d}{d\sigma}(JK)$, where $J(x,y)$ is the geometric ray divergence , we reduce the transfer equation (19) to the following conservation law along the rays [2,13,18-20]

$$\frac{d}{d\sigma}\left(\ln \frac{A_0^2(x,y)J(x,y)}{K^2(x,y)}\right) = 0. \tag{20}$$

The conservation law (20) can be written as well in the form suitable for finding the function A_0 [15,18-20]

$$\frac{A_0^2(x(\sigma,\alpha), y(\sigma,\alpha))}{K^2(x(\sigma,\alpha), y(\sigma,\alpha))} d\,a(x(\sigma,\alpha), y(\sigma,\alpha)) =$$

$$= \frac{A_0^2(x_0(\alpha), y_0(\alpha))}{K^2(x_0(\alpha), y_0(\alpha))} d\,a(x_0(\alpha), y_0(\alpha)), \tag{21}$$

where $d\,a(x(\sigma,\alpha), y(\sigma,\alpha)) = J(x(\sigma,\alpha), y(\sigma,\alpha))d\alpha$ is the unit ray tube width. Note that the wave energy flash is proportional to $A_0^2 K^{-1} da$, thus, from (21) it follows that, in this case, there survives the value equal to the wave energy flash divided by the wave vector modulus [2, 15,17].

Non-harmonic internal gravity wave packets

To proceed to studying the problem of non-harmonic wave packets evolution in a smoothly non-uniform horizontally stratified medium we presuppose the choice of Anzatz (Anzatz is the German a solution type [1,13,14]), which define the propagation of Airy and Fresnel internal waves with certain heuristic arguments.

Airy wave. Let's introduce the slow variables $x^* = \varepsilon x, \quad y^* = \varepsilon y, \quad t^* = \varepsilon t$ (no slowness is supposed over z, the index is omitted hereafter), where $\varepsilon = \lambda / L << 1$ is the small parameter that characterizes the softness of ambient horizontal changes (λ is the typical wave length, L is the scale of a horizontal non-uniformity). Then the system (2) in slow variables will be written as follows:

$$\frac{\partial^4 W}{\partial z^2 \partial t^2} + \varepsilon^2 \frac{\partial^2 W}{\partial t^2} + \frac{g}{\rho_0} \Delta(\varepsilon U_1 \frac{\partial \rho_0}{\partial x} + \varepsilon U_2 \frac{\partial \rho_0}{\partial y} + W \frac{\partial \rho_0}{\partial z}) = 0, \tag{22}$$

$$\varepsilon \Delta U_1 + \frac{\partial^2 W}{\partial z \partial x} = 0 \qquad \varepsilon \Delta U_2 + \frac{\partial^2 W}{\partial z \partial y} = 0.$$

Next we examine the superimposition of harmonic waves (in slow variables x, y, t)

$$W = \int \omega \sum_{m=0}^{\infty} (i\varepsilon)^m W_m(\omega, z, x, y) \exp(\frac{i}{\varepsilon}[\omega t - S_m(\omega, x, y)]) d\omega. \tag{23}$$

With respect to functions $S_m(\omega, x, y)$ it is assumed that they are odd-numbered on ω and $\min_{\omega} \partial S / \partial \omega$ is reached at $\omega = 0$ (for all x and y). Substituting (23) into (22) we can easily have it proved that the function $W_m(\omega, z, x, y)$ has at $\omega = 0$ a pole of the *m*-th order . Therefore, as the model integral $R_m(\sigma)$ for individual terms in (23) will serve the following formulas: $R_m(\sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{i}{\omega}\right)^{m-1} \exp(i\left(\frac{\omega^3}{3} - \sigma\omega\right)) d\omega$, where the integration contour is going around the point $\omega = 0$ from overhead, which enables the functions $R_m(\sigma)$ to exponentially decay at $\sigma >> 1$ [21-22]. The functions $R_m(\sigma)$ have the following feature: $\frac{d R_m(\sigma)}{d\sigma} = R_{m-1}(\sigma)$,at that $R_0(\sigma) = Ai'(\sigma)$, $R_1(\sigma) = Ai(\sigma)$, $R_2(\sigma) = \int_{-\infty}^{\sigma} Ai(u) du$, etc. It is evident, considering certain properties of Airy integrals, that the functions $R_m(\sigma)$ related with each other as: $R_{-1}(\sigma) + \sigma R_1(\sigma) = 0$, $R_{-3}(\sigma) + 2 R_0(\sigma) - \sigma^2 R_1(\sigma) = 0$ [21,22].

Fresnel wave. As the model integrals $R_m(\sigma)$ that describe the propagation of Fresnel waves taking into account the solution structure for the displacement in the horizontally uniform case we use the following formulas: $R_0(\sigma) = \mathrm{Re} \int_0^{\infty} \exp\left(-it\sigma - i\frac{t^2}{2}\right) dt \equiv \mathrm{Re}\,\Phi^*(\sigma) \equiv \Phi(\sigma)$. It's easy to see that function $\Phi^*(\sigma)$ has the following feature

$$\frac{d\,\Phi^*(\sigma)}{d\sigma} = -\int_0^{\infty} it \exp(-it\sigma - \frac{1}{2} i\,t^2) dt =$$

$$= \int_0^\infty \exp(-it\sigma - \frac{1}{2}it^2)d(-it\sigma - \frac{1}{2}it^2) + i\sigma\int_0^\infty \exp(-it\sigma - \frac{1}{2}it^2)dt = 1 + i\sigma\,\Phi^*(\sigma)$$

,

From which we can obtain, for instance: $\frac{d^3\Phi^*(\sigma)}{d\sigma^3} = i\sigma\frac{d^2\Phi^*(\sigma)}{d\sigma^2} + 2i\frac{d\Phi^*(\sigma)}{d\sigma}$,or (in terms of functions $R_m(\sigma)$): $R_{-1}(\sigma) + i\sigma R_0(\sigma) = 0$, $R_{-3}(\sigma) - 2iR_{-1}(\sigma) - i\sigma R_{-2}(\sigma) = 0$ [1,21,22].

Based on the above, and as well on the first member structure of the Airy and Fresnel uniform wave asymptotics for a horizontally uniform medium, the solution of the system in (22) can be found, for instance, in the form (for an individually taken mode W_n, $\mathbf{U}_n$, further omitting the index n) [1,2,15]

$$W = \varepsilon^0 W_0(z,x,y,t)R_0(\sigma) + \varepsilon^a W_1(z,x,y,t)R_1(\sigma) + \varepsilon^{2a}W_2(z,x,y,t)R_2(\sigma) + \ldots \qquad (24)$$

$$\mathbf{U} = \varepsilon^{1-a}\mathbf{U}_0(z,x,y,t)R_1(\sigma) + \varepsilon\mathbf{U}_1(z,x,y,t)R_2(\sigma) + \varepsilon^{1+a}\mathbf{U}_2(z,x,y,t)R_3(\sigma) + \ldots,$$

where the argument $\sigma = \left(\frac{1}{a}S(x,y,t)\right)^a \varepsilon^{-a}$ is assumed to be of the order of unity. Expansion Expression (24) agrees with a common approach of the geometric optics method (spatial time ray method, WKBJ method) [1-3,13,16,17] .

Note also that from such a solution structure it follows that the solution for a horizontally non-uniform and non-stationary medium shall depend on both the "fast" (vertical coordinate) and "slow" (time and horizontal coordinates) variables. Next we generally are going to find a solution in "slow" variables, at that the solution's structural elements which depend on the "fast" variables appear in the form of integrals of some slowly varying functions along the space-time rays.

This solution choice allows us to define the uniform asymptotics for internal gravity wave fields propagating within stratified layer with slowly varying parameters, which holds true either near or far away from the wave fronts of a single wave mode. If we need only to define the behavior of a field near the wave front, then we can use one of the geometric optics methods – the "progressing wave" method, and a weakly dispersive approximation in the form of appropriate local asymptotics, and find the representation for the phase functions argument σ in the form: $\sigma = \alpha(t,x,y)(S(t,x,y) - \varepsilon t)\varepsilon^{-a}$, where the function $S(t,x,y)$ defines the wave front position and is determined from the eikonal equation solution: $\nabla^2 S = c^{-2}(x,y,t)$ [1,2,13,15]. A function $c(t,x,y)$ is the maximum group velocity of a respective wave mode, i.e., the first member of the dispersion curve expansion in zero [1,15]. The function $\alpha(t,x,y)$ (the second member of the expanded dispersion curve) describes the space-time impulse width evolution of Airy or Fresnel non-harmonic waves, and then it will be defined from some arbitrary laws of conservation along the eikonal equation characteristics with their actual form to be determined by the problem physical conditions [1-3, 14].

Eikonal equation: rays and initial conditions formulation

Next we focus on the eikonal equation and formulation of initial settings for it. In this section we shall examine generally two basic eikonal equations and relative characteristic systems for solving these equations. The first type eikonal equation is obtained by constructing the asymptotics within a limited space domain, i.e., near the wave front of a single mode, that is to say we use a "weak dispersion" approximation [1,15]. In this case the dispersion curve, having its properties defined by the wave field near the front, is approximated by appropriate Taylor expansion limited by the third member of the asymptotic series. Then, as it will be illustrated in [1,15], to define the moment of the wave front $S(x, y, t)$ arrival we have the following eikonal equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \equiv p^2 + q^2 = c^{-2}(x, y, t), \qquad (25)$$

$$(p,q) = -\nabla S,$$

where $c(x, y, t)$ is the maximum group velocity of propagating individual internal gravity waves mode .

The solution of the vertical spectrum problem with appropriate boundary conditions

$$\frac{\partial^2 A(x, y, z, t)}{\partial z^2} + \frac{N^2(z, x, y, t)}{c^2(x, y, t)} A(z, x, y, t) = 0$$

is assumed to be known [1-3]. Then for solving the equation of this eikonal (25) the system of characteristic equations will be written as [18-20]

$$\frac{dx}{dt} = c^2(x, y, t)\, p \qquad \frac{dp}{dt} = -\frac{c_x'(x, y, t)}{c(x, y, t)},$$
$$\frac{dy}{dt} = c^2(x, y, t)\, q \qquad \frac{dq}{dt} = -\frac{c_y'(x, y, t)}{c(x, y, t)}. \qquad (26)$$

Hence, we get, for example, the relationship as follows:

$$\frac{dS}{dt} = \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial y}\frac{dy}{dt} = p\,c^2 p + q\,c^2 q = c^2(p^2 + q^2) = 1, \qquad (27)$$

which indicates that for this case the eikonal $S = t - t_0$ is the linear function of time.

As the initial settings for solving the system (26) we may use the data on some line. The problem under investigation is two-dimensional, because it is assumed that the medium parameters are slowly varying in time and horizontally, and no slowness in the vertical coordinate is required. Therefore, let the plane (x, y) have a parametrically defined line: $l = (x_0(\lambda), y_0(\lambda)$, on which we can plot the eikonal initial distribution: $S = S_0(\lambda)$. The physical meaning of setting the initial conditions in such a form is rather evident: the horizontal internal wave falls on the boundary between horizontally uniform and non-uniform mediums, the interface between the mediums is defined by this line, all wave field parameters at this border line are known. Each value of the parameter λ identically determines its value $S_0(\lambda)$ and the Cartesian ordinates $x_0(\lambda), y_0(\lambda)$, from which the characteristics is "produced", that is, the system (26) solution.

In the examined system (26) by reason of its definition the parameter that is varying along the characteristics is the time t. Selected as the initial value is the time of exit of radiation. Let there be a point on the line l from which there originates a characteristics (ray) with parameter λ for which it takes time t to reach an arbitrary point (x, y). Evidently, we may assume that these values (x, y) are the functions of the parameters λ and t, that is, $x = x(\lambda,t), y = y(\lambda,t)$. If the transformation Jacobian from ray coordinates (λ,t) to the Cartesian (x, y) differs from zero then these equations are solvable with respect to variables (λ,t): $\lambda = \lambda(x, y), t = t(x, y)$. The function $t(x, y)$ at $x_0(\lambda), y_0(\lambda)$ is given as $t(x, y) = S_0(\lambda)$, and as the initial value we take the moment (time) of exit for the characteristics (ray). Hence, we have the system

$$\frac{dt}{dt} = 1,$$

$$t(x, y) = S_0(\lambda) \text{ at } x = x_0(\lambda), y = y_0(\lambda).$$

Then, using the relation (27), we obtain that $t(x, y) = S(x, y)$.

The second type eikonal equation results from constructing the proportional asymptotics of the far fields of internal gravity waves. The eikonal equation for determining the phase S of non-harmonic wave packets that describe these proportional asymptotics shall be written as

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \equiv p^2 + q^2 = K^2(\omega, x, y), \tag{28}$$

$$\mathbf{k} \equiv (p,q) = -\nabla S, \quad \omega = \partial S / \partial t .$$

Solving the vertical spectral problem with respective boundary conditions

$$\frac{\partial^2 A(z,x,y,t)}{\partial z^2} + |\mathbf{k}|^2\left(\frac{N^2(z,x,y,t)}{\omega^2} - 1\right)A(z,x,y,t) = 0$$

is also assumed to be known. The dispersion dependency denoted as $K(\omega, x, y)$ is defined from the solution of this vertical spectral problem, ω is the spectral parameter. The equation (28) is the Hamilton-Jacobi equation with Hamiltonian operator $|\mathbf{k}|^2 - K^2(\omega, x, y)$ [18-20], and the characteristic system of this equation is given by

$$\frac{dx}{dt} = \frac{p}{K(\omega,x,y)K'_{\omega}(\omega,x,y)} \quad \frac{dy}{dt} = \frac{q}{K(\omega,x,y)K'_{\omega}(\omega,x,y)}$$

$$\frac{dp}{dt} = \frac{K'_x(\omega,x,y)}{K'_{\omega}(\omega,x,y)} \quad \frac{dq}{dt} = \frac{K'_y(\omega,x,y)}{K'_{\omega}(\omega,x,y)} \quad \frac{d\omega}{dt} = 0 .$$

Consider the characteristic system as follows

$$\frac{dx}{dt} = p(\omega, x, y),$$
$$\frac{dy}{dt} = q(\omega, x, y), \qquad (29)$$
$$\frac{d\omega}{dt} = 0,$$

where the frequency ω and the time of exit of radiation t_0 are the ray coordinates. The solutions of this system are: $x = x(t, t_0, \omega)$, $y = y(t, t_0, \omega)$. The Jacobian of transition from the Cartesian ordinates (x, y) to ray coordinates (t_0, ω) is written as: $D = D(t, t_0, \omega) = \frac{\partial x}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial y}{\partial \omega}\frac{\partial x}{\partial t_0}$. Next we calculate: $\frac{dD}{dt} = \frac{d}{dt}(\frac{\partial x}{\partial \omega})\frac{\partial y}{\partial t_0} + \frac{d}{dt}(\frac{\partial y}{\partial t_0})\frac{\partial x}{\partial \omega} - \frac{d}{dt}(\frac{\partial x}{\partial t_0})\frac{\partial y}{\partial \omega} - \frac{d}{dt}(\frac{\partial y}{\partial \omega})\frac{\partial x}{\partial t_0}$. Using the characteristic system (29) we can obtain

$$\frac{d}{dt}(\frac{\partial x}{\partial \omega}) = \frac{\partial p}{\partial \omega} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial \omega} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \omega},$$
$$\frac{d}{dt}(\frac{\partial y}{\partial \omega}) = \frac{\partial q}{\partial \omega} + \frac{\partial q}{\partial x}\frac{\partial x}{\partial \omega} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial \omega},$$
$$\frac{d}{dt}(\frac{\partial x}{\partial t_0}) = \frac{\partial p}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial t_0},$$
$$\frac{d}{dt}(\frac{\partial y}{\partial t_0}) = \frac{\partial q}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial t_0}.$$

Then we have

$$\frac{dD}{dt} = (\frac{\partial p}{\partial \omega} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial \omega} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \omega})\frac{\partial y}{\partial t_0} + (\frac{\partial q}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial t_0})\frac{\partial x}{\partial \omega} -$$
$$-(\frac{\partial p}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial t_0})\frac{\partial y}{\partial \omega} - (\frac{\partial q}{\partial \omega} + \frac{\partial q}{\partial x}\frac{\partial x}{\partial \omega} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial \omega})\frac{\partial x}{\partial t_0} =$$
$$= \frac{\partial p}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial q}{\partial \omega}\frac{\partial x}{\partial t_0} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial p}{\partial x}\frac{\partial y}{\partial \omega}\frac{\partial x}{\partial t_0} +$$
$$+ \frac{\partial q}{\partial y}\frac{\partial x}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial q}{\partial y}\frac{\partial y}{\partial \omega}\frac{\partial x}{\partial t_0} = \frac{\partial p}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial q}{\partial \omega}\frac{\partial x}{\partial t_0} + (\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y})D.$$

Next we show that there is the relationship

$$\frac{\partial p}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial q}{\partial \omega}\frac{\partial x}{\partial t_0} = (\frac{\partial p}{\partial \omega}\frac{\partial \omega}{\partial x} + \frac{\partial q}{\partial \omega}\frac{\partial \omega}{\partial y})D.$$

Hence, the following equality will be realized

$$\frac{dD}{dt} = divc(\omega, x, y)D,$$
$$c(\omega, x, y) \equiv (\frac{dx}{dt}, \frac{dy}{dt}) = (p, q), \qquad (30)$$

$$divc(\omega,x,y)=(\frac{\partial}{\partial x}+\frac{\partial}{\partial\omega}\frac{\partial\omega}{\partial x},\frac{\partial}{\partial\omega}\frac{\partial\omega}{\partial y})c(\omega,x,y)=$$
$$=\frac{\partial p}{\partial x}+\frac{\partial p}{\partial\omega}\frac{\partial\omega}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial q}{\partial\omega}\frac{\partial\omega}{\partial y}.$$

In the case under investigation the operator $divc(\omega,x,y)$ accounts for the explicit dependence of $c(x,y)=c(\omega(x,y),x,y)$ on the variables (x,y) and the implicit dependence on these variables since it is evident that the frequency $\omega=\omega(x,y)$ is also a function of these variables. Now we calculate on the characteristics (29) the values of functions $\frac{\partial\omega}{\partial x},\frac{\partial\omega}{\partial y}$. Differentiating the equation of characteristics $x=x(t,t_0,\omega)$, $y=y(t,t_0,\omega)$ on variables (x,y) we have consequently

$$\frac{\partial x(t,t_0,\omega)}{\partial x}=1=\frac{\partial x}{\partial t_0}\frac{\partial t_0}{\partial x}+\frac{\partial x}{\partial\omega}\frac{\partial\omega}{\partial x},$$
$$\frac{\partial x(t,t_0,\omega)}{\partial y}=0=\frac{\partial x}{\partial t_0}\frac{\partial t_0}{\partial y}+\frac{\partial x}{\partial\omega}\frac{\partial\omega}{\partial y},$$
$$\frac{\partial y(t,t_0,\omega)}{\partial y}=1=\frac{\partial y}{\partial t_0}\frac{\partial t_0}{\partial y}+\frac{\partial y}{\partial\omega}\frac{\partial\omega}{\partial y},$$
$$\frac{\partial y(t,t_0,\omega)}{\partial x}=0=\frac{\partial y}{\partial t_0}\frac{\partial t_0}{\partial x}+\frac{\partial y}{\partial\omega}\frac{\partial\omega}{\partial x}.$$

From which we obtain the following relationships

$$\frac{\partial\omega}{\partial x}=\frac{\partial y}{\partial t_0}/D,\qquad \frac{\partial\omega}{\partial y}=-\frac{\partial x}{\partial t_0}/D,$$
$$\frac{\partial t_0}{\partial x}=-\frac{\partial y}{\partial\omega}/D,\qquad \frac{\partial t_0}{\partial y}=\frac{\partial x}{\partial\omega}/D.$$

Then we have: $(\frac{\partial p}{\partial\omega}\frac{\partial\omega}{\partial x}+\frac{\partial q}{\partial\omega}\frac{\partial\omega}{\partial y})D=(\frac{\partial p}{\partial\omega}\frac{\partial y}{\partial t_0}/D-\frac{\partial q}{\partial\omega}\frac{\partial x}{\partial t_0}/D)D=\frac{\partial p}{\partial\omega}\frac{\partial y}{\partial t_0}-\frac{\partial q}{\partial\omega}\frac{\partial x}{\partial t_0}$. By this means the relationship (30) has been proved. It is evident that in the space (x,y,p,q) such proving is not necessary, whereas the equality $\frac{d\ln D}{dt}=divc$ in the space (x,y,p,q) is evident by virtue of the operator definition $div=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial p},\frac{\partial}{\partial q})$ within this space. The requirement of proving the relationship (30) comes about from the fact that the characteristics (rays) are not viewed in the four-dimension space (x,y,p,q), but in the two-dimension Cartesian space (x,y) [1, 13, 18-20].

Take a different setting of the initial data for solving the characteristic system in (29), such as the initial data setting at the moment of time $t=t_0$ on the parametrically defined line $(x=x(\sigma),y=y(\sigma))$, (t_0,σ) are the ray coordinates.

Then we have: $\omega = \dfrac{\partial S}{\partial t_0} \equiv \Omega(t_0, \sigma)$ at $t = t_0$, $x = x_0(\sigma)$, $y = y_0(\sigma)$. On this line the eikonal S is the well-known function. Then the characteristics equation can be written as

$$\begin{aligned} &\omega = \Omega(t_0, \sigma), \\ &x = x(t, t_0, \sigma),\ x = x_0(\sigma), \text{ at } t = t_0, \\ &y = y(t, t_0, \sigma),\ y = y_0(\sigma), \text{ at } t = t_0. \end{aligned} \tag{31}$$

For the examined case we write out a formula for the Jacobian transition from the Cartesian ordinates (x, y) to the ray coordinates (t_0, σ): $D = D(t, t_0, \sigma) = \dfrac{\partial x}{\partial \sigma}\dfrac{\partial y}{\partial t_0} - \dfrac{\partial y}{\partial \sigma}\dfrac{\partial x}{\partial t_0}$. Next we calculate

$$\frac{dD}{dt} = \frac{d}{dt}\left(\frac{\partial x}{\partial \sigma}\right)\frac{\partial y}{\partial t_0} + \frac{d}{dt}\left(\frac{\partial y}{\partial t_0}\right)\frac{\partial x}{\partial \sigma} - \frac{d}{dt}\left(\frac{\partial x}{\partial t_0}\right)\frac{\partial y}{\partial \sigma} - \frac{d}{dt}\left(\frac{\partial y}{\partial \sigma}\right)\frac{\partial x}{\partial t_0}. \tag{32}$$

Using the characteristic system (29) and initial data in the form (31), we obtain the following relationships:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial \sigma}\right) = \frac{\partial p}{\partial \omega}\frac{\partial \Omega}{\partial \sigma} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \sigma},$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial \sigma}\right) = \frac{\partial q}{\partial \omega}\frac{\partial \Omega}{\partial \sigma} + \frac{\partial q}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial \sigma},$$

$$\frac{d}{dt}\left(\frac{\partial x}{\partial t_0}\right) = \frac{\partial p}{\partial \omega}\frac{\partial \Omega}{\partial t_0} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial t_0},$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial t_0}\right) = \frac{\partial q}{\partial \omega}\frac{\partial \Omega}{\partial t_0} + \frac{\partial q}{\partial x}\frac{\partial x}{\partial t_0} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial t_0}.$$

Using these relationships we can calculate

$$\frac{dD}{dt} = \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)D + \left(\frac{\partial \Omega}{\partial \sigma}\frac{\partial y}{\partial t_0} - \frac{\partial \Omega}{\partial t_0}\frac{\partial y}{\partial \sigma}\right)\frac{\partial p}{\partial \omega} + \left(-\frac{\partial \Omega}{\partial \sigma}\frac{\partial x}{\partial t_0} + \frac{\partial \Omega}{\partial t_0}\frac{\partial x}{\partial \sigma}\right)\frac{\partial q}{\partial \omega}.$$

Later differentiating the initial conditions (31) on (x, y) respectively, and by repeating the above algorithm we can get

$$\frac{\partial \omega}{\partial x} = \frac{\partial \Omega}{\partial t_0}\frac{\partial t_0}{\partial x} + \frac{\partial \Omega}{\partial \sigma}\frac{\partial \sigma}{\partial x} = \frac{\partial \Omega}{\partial t_0}\left(-\frac{\partial y}{\partial \sigma}/D\right) + \frac{\partial \Omega}{\partial \sigma}\left(\frac{\partial y}{\partial t_0}/D\right),$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial \Omega}{\partial t_0}\left(\frac{\partial x}{\partial \sigma}/D\right) - \frac{\partial \Omega}{\partial \sigma}\left(\frac{\partial x}{\partial t_0}/D\right).$$

Hence, resulting from (32) we also come to the formula (30).

We have to make a few points with regard to the Jacobian geometric meaning of the discussed problem in relation to non-harmonic packets of internal gravity waves generated by disturbing sources in motion [1,2,17]. Consider a characteristic system for the horizontally uniform stationary scenario and a point disturbing source in motion with velocity V. The equations of characteristics (rays) are defining the straight lines

$$x(t) = Vt_0 + \frac{\omega(t - t_0)}{VK(\omega)K'_\omega(\omega)},$$

$$y(t) = \frac{\nu(\omega)(t - t_0)}{K(\omega)K'_\omega(\omega)},$$

$$\nu(\omega) = \sqrt{K^2(\omega) - p^2} = \sqrt{K^2(\omega) - (\omega / V)^2}.$$

The Jacobian transition from the ray coordinates (t_0, ω) to Cartesian ordinates (x, y) is given by

$$D = \frac{\partial x}{\partial \omega}\frac{\partial y}{\partial t_0} - \frac{\partial y}{\partial \omega}\frac{\partial x}{\partial t_0} =$$

$$= \frac{\partial}{\partial \omega}\left(\frac{\omega}{VK(\omega)K'_\omega(\omega)}\right)\frac{\nu(\omega)}{K(\omega)K'_\omega(\omega)} - \frac{\partial}{\partial \omega}\left(\frac{\nu(\omega)}{VK(\omega)K'_\omega(\omega)}\right)\left(V + \frac{\omega}{VK(\omega)K'_\omega(\omega)}\right)$$

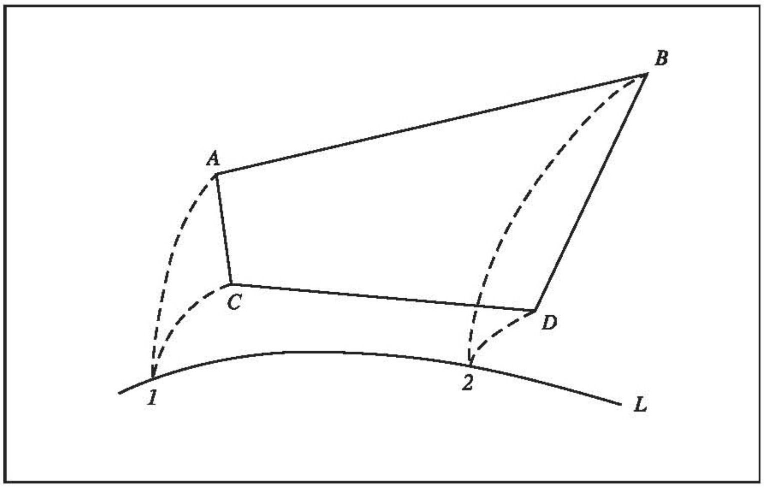

Fig. 1. The geometrical meaning of the Jacobian. L is the source motion trajectory within the Cartesian space; 1A is the characteristic (ray) with a constant value of ray coordinates $(\omega,\ t_0)$*; 1C is the characteristic (ray) with a constant value of ray coordinates* $(\omega + \Delta\omega,\ t_0)$*; 2B is the characteristic (ray) with a constant value of ray coordinates* $(\omega,\ t_0 + \Delta t_0)$*; 2D is the characteristic (ray) with a constant value of ray coordinates* $(\omega + \Delta\omega,\ t_0 + \Delta t_0)$*.*

Then, on account of the geometric construction shown in Fig.1 it is evident that in the first approximation the area Σ of the surface element *ABCD* equals: $\Sigma = D\Delta\omega\Delta t_0$. Here lines *1A*(*1C*) and *2B*(*2D*) are the constant value lines, that is, the contour lines of ray variables ω and t_0 in the Cartesian space (x, y). It is evident that in non-dispersive mediums because of no divergence of characteristics (rays) the area size Σ of the surface element *ABCD*, by virtue of its definition is equal to zero. If, apart from the dispersion the stratified medium is also horizontally non-uniform, then the characteristics (rays) are not straight lines, the surface element *ABCD* only in the first approximation and at quite small $\Delta\omega$ and Δt_0 is approximated by a respective parallelogram. For the case of a horizontally uniform medium the characteristics (rays) are always straight lines and the element *ABCD* is always a parallelogram. Thus, the Jacobian of ray coordinates transiting to the Cartesian ordinates outlines the geometric divergence in the Cartesian space of the characteristics (rays) for the respective eikonal equation [1,2,3,18-20].

Now we go back to the two principal methods of defining the initial settings for solving the characteristic system of an arbitrary eikonal equation: $F(p,q,x,y,\omega)=0$, $(p,q)\equiv-\nabla S$, $\omega=\partial S/\partial t$, which determines the phase function $S(x,y,t)$ [1,13,15,18-20]. The first method consists in that we define an arbitrary initial space-time domain $(x,y,t): x=x_0(\lambda,\sigma)\ y=y_0(\lambda,\sigma)\ t=t_0(\lambda,\sigma)$, where λ,σ are some ray variables, in which all initial data is defined, including the phase function: $S(x,y,t)=S(x_0,y_0,t_0)=S_0(\lambda,\sigma)$. The characteristics (rays) are «let out» of this space-time domain which is the beginning of further dynamics for the characteristics (rays). It is evident that the following relationships are holding for this domain

$$\frac{\partial S_0}{\partial\lambda}=\frac{\partial S_0}{\partial x_0}\frac{\partial x_0}{\partial\lambda}+\frac{\partial S_0}{\partial y_0}\frac{\partial y_0}{\partial\lambda}+\frac{\partial S_0}{\partial t_0}\frac{\partial t_0}{\partial\lambda}=p_0\frac{\partial x_0}{\partial\lambda}+q_0\frac{\partial y_0}{\partial\lambda}+\omega_0\frac{\partial t_0}{\partial\lambda},$$

$$\frac{\partial S_0}{\partial\sigma}=\frac{\partial S_0}{\partial x_0}\frac{\partial x_0}{\partial\sigma}+\frac{\partial S_0}{\partial y_0}\frac{\partial y_0}{\partial\sigma}+\frac{\partial S_0}{\partial t_0}\frac{\partial t_0}{\partial\sigma}=p_0\frac{\partial x_0}{\partial\sigma}+q_0\frac{\partial y_0}{\partial\sigma}+\omega_0\frac{\partial t_0}{\partial\sigma}.$$

Using the eikonal equation for the initial values

$$F(p_0,q_0,x_0,y_0,\omega_0)=0 \tag{33}$$

we obtain three equations to determine the remaining unknown initial values p_0,q_0,ω_0 within this initial space-time domain.

Us a rule the physical statement of a problem related to non-harmonic packets of internal gravity waves generated in real-world natural environments involves the following. There are two mediums: a horizontally uniform and a horizontally non-uniform, which have a certain interface (border line) in between the form of which is well-known [1,2,15]. On this border line based the problem solution for a uniform medium we assume as known all properties of the examined wave field, that is, the phase function $S(x,y,t)$ is considered as known for all moments of the time t. Accounting for that, we have to define the evolution of non-harmonic wave packets in a horizontally non-uniform medium. To this end it seems more convenient to set the initial data for solving a system of characteristics in other format, that is, in the form of arbitrary "boundary" conditions as defined on this border line between two mediums. What this means is that we define the line $x=x_0(\alpha), y=y_0(\alpha)$ at $t=t_0$, over which the phase function is well-known at any moment of time $t: S(x,y,t)=S_0(\alpha,t,t_0)$. It's worth to mention that the boundary line between two mediums is not time dependent. On this line evidently we can by virtue of definition realize the following par: $\omega_0 = \frac{\partial S_0}{\partial t_0}$. Then to determine the initial values p_0, q_0 on this line we have

$$\frac{\partial S_0}{\partial \alpha} = \frac{\partial S_0}{\partial x_0}\frac{\partial x_0}{\partial \alpha} + \frac{\partial S_0}{\partial y_0}\frac{\partial y_0}{\partial \alpha} = p_0\frac{\partial x_0}{\partial \alpha} + q_0\frac{\partial y_0}{\partial \alpha},$$

which together with the relationship (33) allows us to define the initial data values for these two variables. In what follows we shall apply this method to formulate the initial data for solving a characteristic system [1,15,17,18-20].

Asymptotical analysis of stratified non-stationary medium wave dynamics

Under the real oceanic conditions the Vaisala-Brunt frequency $N^2(z,t) = -g\partial \ln \rho / \partial z$ defines the basic characteristics of internal gravity wave dynamic, and it shall not depend solely on spatial variables *(x,y,z)*, but also on the time t. The most characteristic types of $N^2(z,t)$ time-to-time variability are the thermocline going up or down and changing its width, etc [4,7,9]. There is a number of time scales for variations of hydro-physical fields in the oceans and seas: a small-scale with periods of about 10 minutes, a meso-scale with periods of about a day (twenty-four hours), as well as synoptical and global variations with periods of a few months to a few years [5,11,12]. In what follows we shall analyze the internal gravity field propagation in non-stationary mediums with parameter variation periods of a day and over, which allows us to use the geometric optics approximation because the period of internal gravity waves is tens of minutes and less. The system of linearized equations of hydrodynamics, when the non-perturbated density ρ depends on variables z and t, reduces to a single equation, for example, for the vertical velocity component: $\left(\frac{\partial}{\partial t} + \frac{\partial \ln \rho}{\partial t}\right)\left[\frac{\partial}{\partial t}\left(\Delta + \frac{\partial^2}{\partial z^2}\right) + \frac{\partial \ln \rho}{\partial z}\frac{\partial^2}{\partial t \partial z}\right]W = g\frac{\partial \ln \rho}{\partial z}\Delta W$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If $\partial \ln \rho / \partial z$ is neglected, we obtain an equation in the Boussinesq approximation: $\left(\frac{\partial}{\partial t} + \frac{\partial \ln \rho}{\partial t}\right)\left[\frac{\partial}{\partial t}\left(\Delta + \frac{\partial^2}{\partial z^2}\right)\right]W + N^2(z,t)\Delta W = 0$. It appears natural to neglect as well the member with $\partial \ln \rho / \partial t$, which would correspond to a consequent application of the Boussinesq hypothesis [1-3]. It means the density characterizing the fuids's inert mass can be assumed constant. Then we have: $\frac{\partial^2}{\partial t^2}(\Delta + \frac{\partial^2}{\partial z^2})W + N^2(z,t)\Delta W = 0$. The resulting equation differs from a standard equation of internal gravity waves in a stationary stratified medium just by the time t parametrical inclusion into the Vaisala-Brunt frequency.

The asymptotic solution is found in the form of a sum of modes with every one of them propagating independently of each other (the adiabatic approximation). We are going to examine a single individually taken mode while omitting its index. Next we focus solely on the space region near the wave front which means that we consider the time t as being close to the arrival time of the wave front, henceforth denoted by τ, i.e., we use a weakly dispersive approximation. Consider the wave propagation in a layer of stratified medium $-H < z < 0$ with the Vaisala-Brunt frequency $N^2(z,t)$. We shall seek the solution with boundary conditions $W = 0, \quad z = 0, -H$ in the form

$$W = W_0 + W_1 + O(\varepsilon^{2p}),$$

$$W_0 = (A(\varepsilon x, \varepsilon y, \tau, z) + \frac{\partial A(\varepsilon x, \varepsilon y, \tau, z)}{\partial \tau}(\varepsilon t - \tau) + \cdots)F_0(\varphi),$$

$$F_{m+1}'(\varphi) = F_m(\varphi), \quad \tau = \tau(\varepsilon x, \varepsilon y),$$

$$W_1 = (B(\varepsilon x, \varepsilon y, \tau, z) + \frac{\partial B(\varepsilon x, \varepsilon y, \tau, z)}{\partial \tau}(\varepsilon t - \tau) + \cdots)F_1(\varphi),$$

where $p = 2/3$, $F_0(\varphi) = Ai'(\varphi)$ is the Airy derivative having its argument $\varphi = \alpha(\varepsilon x, \varepsilon y)(\varepsilon t - \tau(\varepsilon x, \varepsilon y))\varepsilon^{-p}$ of the order of unit. The function τ defines the wave front position, function α describes the evolution of the Airy wave width, the small parameter ε specifies "slow variables". Since our focus is only on "slow times" εt being close to the time of the wave front arrival τ, then all functions preceding functions F_m, are given in the form of Taylor series by $\varepsilon t - \tau \approx \varepsilon^p$ powers. Let $N^2(z,\varepsilon t)$ be written as: $N^2(z,\varepsilon t) = N^2(z,\tau) + \frac{\partial N^2(z,\tau)}{\partial \tau}(\varepsilon t - \tau) + O(\varepsilon^{2p})$. We consider the far internal gravity waves generated by moving source. After complicated analytical calculations, we can obtain an expression for the term W_0 in the form:

$$W_0 = \frac{c^{5/2}(\tau_0)\alpha^2(x,y)f(z,\tau)}{2c^{3/2}(\tau)R^{1/2}(x,y,t)(V^2 - c^2(\tau_0))^{1/4}} \frac{\partial f(z_0,\tau_0)}{\partial z_0} Ai'\left(\alpha(x,y)\frac{t-\tau(x,y)}{\varepsilon^{2/3}}\right),$$

where *c, f* - eigenvalue and eigenfunction of internal gravity vertical spectral problem, *V*- source speed, z_0 - depth of source motion, *R(x,y,t)* - some function that are determined by the parameters of the problem [1,2,15,17] .

The figures demonstrate the numerical results of internal gravity wave calculations for typical oceanic parameters [5,6,9,12]. The Fig.2 shows a system of rays (thin line), caustics (bold line) generated by source moving in a non-stationary stratified ocean. It is a general rule that caustic of a family of rays single out an area in space, so that rays of that family cannot appear in the marked area. There is also another area, and each point of that area has two rays that pass through this point. One of those rays has already passed this point, and another is going to pass the point. Formal approximation of geometrical optics or WKBJ approximation cannot be applied near the caustic, that is because rays merge together in that area, after they were reflected by caustic. If we want to find wave field near the caustic, then it is necessary to use special approximation of the solution, and in the paper a modified ray method is proposed in order to build uniform asymptotic expansion of integral forms of the internal gravity wave field. After the rays are reflected by the caustic, there appears a phase shift. It is clear that the phase shift can only happen in the area where methods of geometrical optics, which were used in previous sections, can't be applied. If the rays touch the caustic several times, then additional phase shifts will be added. Phase shift, which was created by the caustic, is rather small in comparison with the change in phase along the ray, but this shift can considerably affect interference pattern of the wave field.

The Fig.3 demonstrates an evolution of internal gravity wave packet in a non-stationary stratified ocean within the system of coordinates that is in motion together with the disturbing source. Time interval for calculations is equal 2 hours. At that it's evident that if there were no Vaisala-Brunt time-to-time frequency variations such a wave coordinate would be stationary. Numerical results show that internal gravity waves dynamic in the real ocean is substantially influenced by non-stationarity of hydro-physical fields. The obtained asymptotic solutions are uniform and allow far internal gravity wave fields to be described both near and far from turning points and wave fronts.

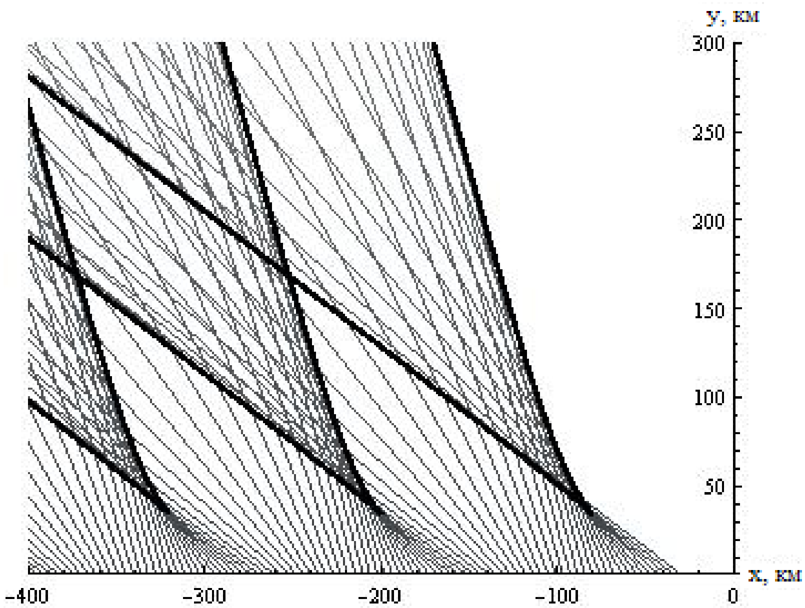

Fig. 2. Rays and caustics in stratified non-stationary medium.

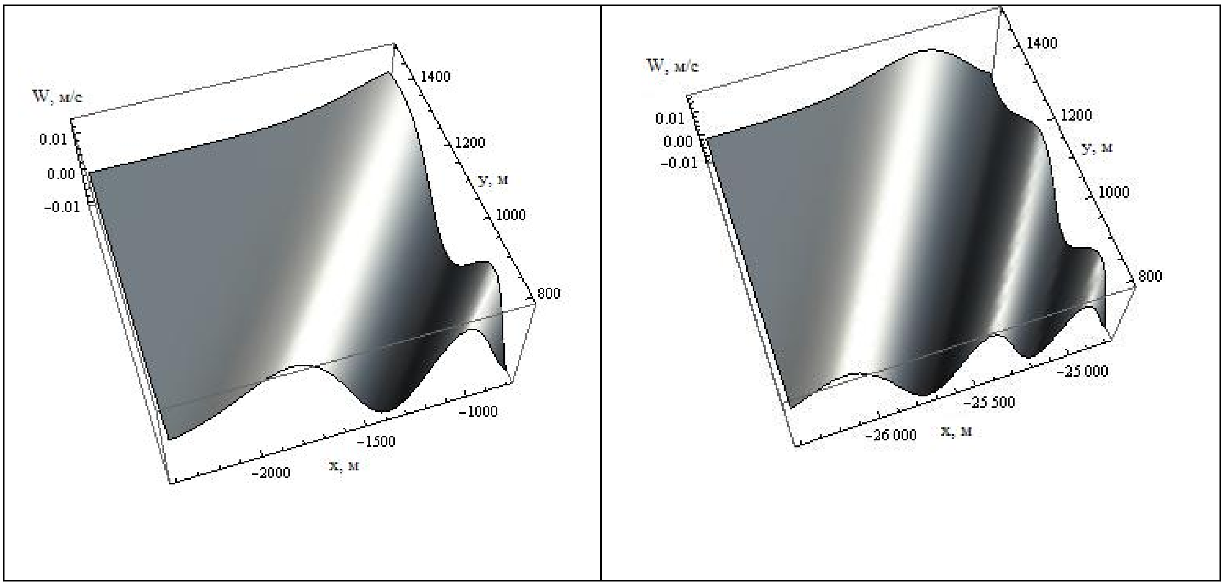

Fig 3. Evolution of internal gravity wave packet in non-stationary stratified medium.

Discussion

Thus, we have the following scheme for far internal gravity waves asymptotic calculating in heterogeneous and non-stationary stratified medium:

1) for an arbitrary density distribution internal gravity waves vertical spectral problems is solved numerically by a shooting method and the corresponding normalized eigenfunctions and eigenvalues are obtained [1,15]
2) eikonal equation is solved numerically along these characteristics (rays) with appropriate initial conditions [1,13-15]
3) after determinations of characteristics (rays) eikonal (phase value) phase functions is calculated by numerical integration along these rays [1-3]
4) geometric divergence ray tubes is determined, for example, numerical differentiation of closely spaced rays [13, 15-17]
5) internal gravity waves amplitude is calculated from the corresponding conservation laws (energy conservation) along the rays (characteristics) [1-3, 13, 15-17].

In this paper a modified space-time ray method is proposed, which belongs to the class of geometrical optics methods (WKBJ approximation). The key point of the proposed technique is the possibility to derive the asymptotic representation of the solution in terms of a non-integer power series of the small parameter

$\varepsilon = \Lambda / L$, where Λ is the characteristic wave length, and L is the characteristic scale of the horizontal heterogeneity. The explicit form of the asymptotic solution was determined based on the principles of locality and asymptotic behavior of the solution in the case of a stationary and horizontally homogeneous medium. The wave packet amplitudes are determined from the energy conservation laws along the characteristic curves. A typical assumption made in studies on the internal wave evolution in stratified media is that the wave packets are locally harmonic. A modification of the geometrical optics method, based on an expansion of the solution in model functions, allows one to describe the wave field structure both far from and at the vicinity of the wave front. The universal character of the asymptotic method proposed for modeling internal gravity fields makes it possible to effectively calculate wave fields and, in addition, qualitatively analyze the obtained solutions. This method offers broad opportunities for the analysis of wave fields on a large scale, which is important **for developing correct** mathematical models of wave dynamics and for assessing in situ measurements of wave fields in the ocean.

The particular role of the proposed asymptotic methods is determined by the fact that the parameters of natural stratified media are usually known approximately and attempts at their adequate numerical solution using the initial equations of hydrodynamics and such parameters may result in a notable loss of accuracy for the results obtained. In addition to their fundamental significance, the obtained asymptotic models are also important for applied investigations, since the proposed method of geometrical optics allows solution of a wide spectrum of problems related to modeling wave fields. In such a situation, the description and analysis of wave dynamics may be realized through developing asymptotic models and using analytical methods for their solution based on the proposed WKBJ modified method.

The results of this work represent a significant interest for physics, mathematics and engineers. Besides that interest analytical, asymptotic and numerical solutions, which were obtained in this paper, can present significant importance for engineering applications, since presented method which were to calculate the internal gravity waves field, make it possible to calculate different wave fields in the rather big class of another problems.

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