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Uniform Asymptotics of the Far Fields of the Surface Disturbances Produced by a Source in a Heavy Infinite-Depth Fluid

V. V. Bulatov, Yu. V. Vladimirov, and I. Yu. Vladimirov

Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, pr. Vernadskogo 101, Moscow, 119526 Russia

e-mail: internalwave@mail.ru

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Abstract—The problem of constructing uniform asymptotics of the far fields of the surface disturbances produced by a localized source in a heavy homogeneous infinite-depth fluid is considered. The solutions obtained govern the wave disturbances both inside and outside the Kelvin wave wedge and are expressed in terms of the Airy function and its derivatives. The results of the numerical calculations of the wave patterns are presented.

Keywords: heavy fluid, surface disturbances, Kelvin wedge, far fields, uniform asymptotics.

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The state of the free surface of the Ocean is influenced by both inhomogeneities occurring within the water thickness, such as obstacles in flow and variations in the bottom relief and in the density and flow fields, and different disturbance sources [1, 2]. To correctly interpret the data obtained in remote sensing of the sea surface, the reasons for some or other surface phenomena must be known. At present, the problem of studying the surface oscillation processes in a density-inhomogeneous and unsteady sea medium remains topical, together with the correlation between the simulation results and visible rough waters. To describe in detail a wide range of physical phenomena connected with the dynamics of surface disturbances of inhomogeneous and unsteady natural media fairly developed mathematical models should be invoked. Usually, these models are sophisticated, nonlinear and dependent on many parameters, so that they can be completely investigated only using numerical methods.

However, in certain cases the primary qualitative idea of the range of phenomena under consideration can be obtained on the basis of simpler asymptotic models and analytical methods for their investigation. Then these models and methods enter in the set of "blocks" of which the complete representation of the wave dynamics consists [6–8]. In this connection, it is necessary to notice the classical fluid mechanics problems on the construction of asymptotic solutions governing the evolution of surface disturbances excited by localized sources in a heavy homogeneous fluid [8–11]. The model solutions thus constructed make it possible to obtain, using the means of the computer mathematics, the asymptotic representations of the wave fields with account for the variability and unsteadiness of the actual natural media [8, 9, 11].

In this study, we consider the problem of the construction of uniform asymptotics of the far fields of the surface disturbances excited by a localized source in a heavy homogeneous infinite-depth fluid.

1. FORMULATION OF THE PROBLEM AND INTEGRAL FORM OF THE SOLUTION FOR THE FREE SURFACE ELEVATION

We will consider the steady pattern of wave disturbances on the surface of the flow of an ideal heavy infinite-depth fluid moving at a velocity V in the positive direction of an x axis. The waves are generated by a point source located at a depth H (the z axis is directed upwards from the undisturbed fluid), the source intensity increasing in accordance with the $q = e^{\varepsilon t}$ law ($-\infty < t < \infty$). In the solution obtained the limit, as $\varepsilon \to 0$, is sought. In view of the problem linearity, the results for the source of arbitrary intensity Q ($Q = \text{const}$) can be obtained by multiplying the result obtained for the source of the unit intensity q by Q (as $\varepsilon \to 0$).

The disturbance of the potential $\Phi(x, y, z, t)$ with respect to the uniform flow moving at a velocity V ($\nabla\Phi = (u, v, w)$, where u, v, and w are the disturbances of the $(V, 0, 0)$ vector), is described by an equation with the corresponding linearized boundary condition on the fluid surface [3–5]

$$\Delta\Phi(x, y, z, t) = e^{\varepsilon t}\delta(x)\delta(y)\delta(z + H), \qquad z < 0,$$
$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \Phi + \frac{\partial\Phi}{\partial z} = 0, \qquad z = 0. \tag{1.1}$$

Here, Δ is the three-dimensional Laplace operator and $\delta(x)$ is the Dirac delta function. The elevation of the free surface of a heavy fluid $Z(x, y, t)$ is related with the potential $\Phi(x, y, z, t)$ by the condition [3, 4]

$$Z(x, y, t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\Phi(x, y, z, t), \qquad z = 0. \tag{1.2}$$

The solution of problem (1.1) is sought in the form $\Phi(x, y, z, t) = e^{\varepsilon t}\varphi(x, y, z)$, where the function $\varphi(x, y, z)$ is determined from the problem

$$\Delta\varphi(x, y, z) = e^{\varepsilon t}\delta(x)\delta(y)\delta(z + H), \qquad z < 0,$$
$$\left(\varepsilon + V\frac{\partial}{\partial x}\right)^2 \varphi + g\frac{\partial\varphi}{\partial z} = 0, \qquad z = 0.$$

The Fourier image of the potential $\varphi(x, y, z)$

$$\Omega(\mu, \nu, z) = \int_{-\infty}^{\infty} e^{i\mu x}dx \int_{-\infty}^{\infty} e^{i\nu y}\varphi(x, y, z)\,dy$$

is determined from the boundary value problem ($k^2 = \mu^2 + \nu^2$)

$$\frac{\partial^2\Omega}{\partial z^2} - k^2\Omega = \delta(z + H), \qquad z < 0,$$
$$(\varepsilon - i\mu V)^2\Omega + g\frac{\partial\Omega}{\partial z} = 0, \qquad z = 0,$$
$$\Omega \to 0, \qquad z \to -\infty,$$

whose solution at z=0 takes the form:

$$\Omega(\mu, \nu, 0) = \frac{-g\exp(-kH)}{(\varepsilon - i\mu V)^2 + gk}.$$

Then, in view of Eq. (1.2), the elevation $\eta(x, y)$ ($Z(x, y, t) = e^{\varepsilon t}\eta(x, y)$) can be represented in the form:

$$\eta(x, y) = \frac{iV}{4\pi^2}\int_{-\infty}^{\infty} e^{-i\nu y}\,dy\int_{-\infty}^{\infty}\frac{\mu\exp(-kH - i\mu x)\,d\mu}{\mu^2V^2 + 2i\varepsilon\mu V - gk}. \tag{1.3}$$

In expression (1.3) the parameter ε is retained only in one term of the denominator, which is necessary for determining the displacement of the pole of the integrand relative to the real axis (into the upper or the lower half-plane).

2. CONSTRUCTION OF THE NONUNIFORM ASYMPTOTICS OF THE SOLUTION: INTEGRATION USING RESIDUES AND THE STATIONARY PHASE METHOD

In the polar coordinates ($x = r\cos\alpha$, $y = r\sin\alpha$), ($\mu = k\cos\psi$, $\nu = k\sin\psi$) expression (1.3) can be brought into the form:

$$\eta(r, \alpha) = \frac{iV}{4\pi^2}\int_0^{2\pi}\cos\psi\,d\psi\int_0^{\infty}\frac{k\exp(-kH - ikr\cos(\psi - \alpha))}{kV^2\cos^2\psi + 2i\varepsilon V\cos\psi - g}\,dk. \tag{2.1}$$

In what follows we will study the expression for the elevation $\eta(r, \alpha)$ at large values of r (to be more precise, for $gr/V^2 \gg 1$). The integrand corresponding to the integration variable k has a simple pole $k^* = gA^{-2} - 2i\varepsilon A^{-1}$, $A = V\cos\psi$ which at $\cos\psi < 0$ is displaced into the upper and at $\cos\psi > 0$ into the lower half-plane. The function $\eta(r, \alpha)$ can be represented as the sum of two terms: $\eta(r, \alpha) = \eta_1(r, \alpha) + \eta_2(r, \alpha)$, the integration with respect to ψ being performed in the region, where $\cos\psi < 0$, for the term $\eta_1(r, \alpha)$ and in the region, where $\cos\psi > 0$, for the term $\eta_2(r, \alpha)$.

To calculate the term $\eta_1(r, \alpha)$ at $\cos(\psi - \alpha) < 0$ the contour of the integration with respect to k can be rotated by $\pi/2$ and coincided with the positive direction of the imaginary axis in the complex plane k (the residue at the pole k^* is taken into account), while at $\cos(\psi - \alpha) > 0$ the integration contour is rotated by $-\pi/2$; in this case, it coincides with the negative direction of the imaginary axis and the residue is not taken into account. It can be shown that in both cases the integral along the imaginary axis is of the order of $O(1/r^2)$, as $r \to \infty$. As a result, the term $\eta_1(r, \alpha)$ takes the form:

$$\eta_1(r, \alpha) = -\frac{g}{2\pi V^3}\int_{\pi/2+\alpha}^{3\pi/2}\cos^{-3}\psi\exp\left(-\frac{gH}{V^2\cos^2\psi}\right)d\psi + O\left(\frac{1}{r^2}\right).$$

The term $\eta_2(r, \alpha)$, which is complex-conjugate with $\eta_1(r, \alpha)$, is determined in a similar fashion. Finally, the expression for the elevation $\eta(r, \alpha)$ can be represented in the form:

$$\eta(r, \alpha) = -\frac{g}{\pi V^3}\int_{\pi/2+\alpha}^{3\pi/2}\cos^{-3}\psi\exp\left(-\frac{gH}{V^2\cos^2\psi}\right)\cos(rS(\psi, \alpha)\,d\psi,$$
$$S(\psi, \alpha) = \frac{g\cos(\psi - \alpha)}{V^2\cos^2\psi}. \tag{2.2}$$

Integral (2.2) is an even function of α. Then at large values of r the asymptotics of integral (2.2) can be calculated using the stationary phase method; for this purpose, the stationary phase points, that is, the roots of the equation $S'_\psi(\psi, \alpha) = 0$ must be determined. On the integration interval $(-\pi/2 + \alpha, 3\pi/2)$ there are two stationary points, namely, $\psi_1(\alpha) = \pi/2 + \alpha/2 + b$ and $\psi_2(\alpha) = \pi + \alpha/2 - b$, ($\psi_1(\alpha) <$

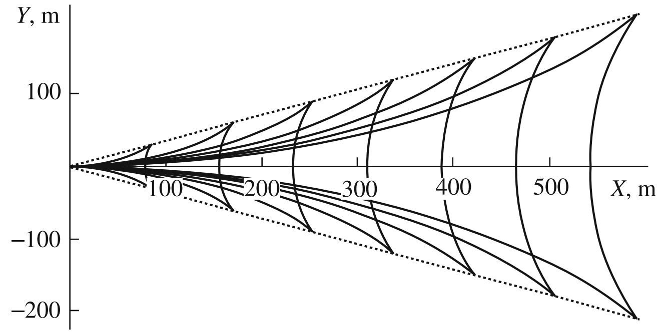

Fig. 1. Phase contours (phase difference between the crests is 2π).

$\psi_2(\alpha)$) at $\alpha > 0$ and $\psi_1(\alpha) = 3\pi/2 + \alpha/2 + b$ and $\psi_2(\alpha) = \pi + \alpha/2 - b$ ($\psi_1(\alpha) > \psi_2(\alpha)$) at $\alpha < 0$; here, $b = \arcsin(3\sin\alpha)/2$. The stationary points exist only for the values of α lying on the interval $-\arcsin(1/3) < \alpha < \arcsin(1/3)$. This condition determines a region on the heavy fluid surface, where wave motions or the Kelvin wedge, can exist [3–5].

The stationary points being known, the well-known pattern of the "ship wave" crests within the Kelvin wedge can readily be determined in the polar coordinates: $r = -2\pi n/S(\psi_1(\alpha), \alpha)$ and $r = -2\pi n/S(\psi_2(\alpha), \alpha)$, $n = 1, 2, 3, \ldots$ The former equality (the stationary point $\psi_1(\alpha)$) is responsible for the longitudinal crests and the latter (the stationary point $\psi_2(\alpha)$) for the transverse crests. The minus sign is taken because at the stationary points the phase is negative.

In Fig. 1 the phase contours (crests) are plotted; the phase difference between the neighboring crests is 2π. Here and in what follows, the calculation parameters are characteristic of the actual oceanic conditions: $V = 11$ m/s and $H = 6$ m [1, 2].

The asymptotics of integral (2.2), as $r \to \infty$, calculated using the stationary phase method, are as follows:

$$\eta(r, \alpha) = -\frac{g}{V^3}\sum_{j=1}^{r}\sqrt{\frac{2}{\pi r|S''_{\psi\psi}(\psi_j, \alpha)|}}\cos^{-3}\psi_j \exp\left(-\frac{gH}{V^2\cos^2\psi_j}\right)\cos\Lambda, \tag{2.3}$$

$$\Lambda = rS(\psi_j, \alpha) + \frac{\pi}{4}\operatorname{sign} S''_{\psi\psi}(\psi_j, \alpha),$$

$$\operatorname{sign} S''_{\psi\psi}(\psi_1, \alpha) = +1, \qquad \operatorname{sign} S''_{\psi\psi}(\psi_2, \alpha) = -1.$$

The nonuniform asymptotics (2.3) work only within the wedge at $-\alpha^* < \alpha < \alpha^*$ (on this interval there are the stationary points) and no longer works as the wedge boundary is approached, where the stationary points coalesce, that is, at $\alpha = \alpha^* = \arcsin 1/3$: $\psi_1(\alpha^*) = \psi_2(\alpha^*) = \alpha^*/2 + 3\pi/4$ and $S_{\psi\psi}(\psi_{1,2}(\alpha), \alpha) \to 0$, as $\alpha \to \alpha^*$.

3. CONSTRUCTION OF THE UNIFORM ASYMPTOTICS OF THE SOLUTION

The uniform asymptotics of integral (2.2) must describe the elevation $\eta(r, \alpha)$ not only inside the Kelvin wave wedge but also outside it and on its boundary. Within the wedge the uniform asymptotics must coincide with the nonuniform asymptotics (2.3) obtained using the stationary phase method. Since the wave pattern for the elevation $\eta(r, \alpha)$ is symmetric about the x axis, that is, $\eta(r, \alpha)$ is an even function of the variable α, in what follows we will consider the case in which $\alpha > 0$.

As shown above, the phase function $S(\psi, \alpha)$ has two turn points $\psi_1(\alpha)$ and $\psi_2(\alpha)$ which coalesce with one another, as $\alpha \to \alpha^* = \arcsin(1/3)$: $\psi_1(\alpha^*) = \psi_2(\alpha^*) = 3\pi/4 + \alpha^* = \psi^*$. Thus, for constructing the uniform asymptotics it is necessary to solve the classical problem of the asymptotics of an integral with two coalescing turn points. Following [6, 7, 12] we can represent integral (2.2) in the form:

$$\eta(r,\ \alpha) = \int\limits_{\pi/2+\alpha}^{3\pi/2} f(\psi)\cos(rS(\psi,\ \alpha))\,d\psi, \tag{3.1}$$

$$f(\psi) = -g\exp(-ghA^{-2})/\pi A^3.$$

Then we make the implicit change of the integration variable

$$S(\psi,\ \alpha) = a_0 + \sigma s - \frac{s^3}{3}. \tag{3.2}$$

In this case, the stationary point $\psi_1(\alpha)$ is associated with the point $s_1 = -\sqrt{\sigma}$ and the point $\psi_2(\alpha)$ with $s_2 = \sqrt{\sigma}$. Then from Eq. (3.2) we obtain

$$a_0(\alpha) = \frac{S(\psi_1) + S(\psi_2)}{2}, \qquad \sigma(\alpha) = \left(\frac{3}{4}(S(\psi_2) - S(\psi_1))\right)^{2/3}. \tag{3.3}$$

After the change (3.2) integral (3.1) takes the form:

$$\eta(r,\ \alpha) = \int\limits_{-\infty}^{\infty} G(s)\cos(r(a_0 + \sigma s - s^3/3))\,ds, \tag{3.4}$$

$$G(s) = f(\psi)\frac{d\psi}{ds}.$$

The lower limit of integral (3.4) is actually determined from Eq. (3.2) at $\psi = \pi/2 + \alpha$ and is the root of the equation $a_0(\alpha) + \sigma(\alpha)s - s^3/3 = 0$. This equation has a unique real root $s^*(\alpha)$ which can be determined in the explicit form using the Cardano formula

$$s^*(\alpha) = \left(\frac{3}{4}\right)^{1/3}\left[(\sqrt{S(\psi_2)} + \sqrt{S(\psi_1)})^{2/3} + (\sqrt{S(\psi_2)} - \sqrt{S(\psi_1)})^{2/3}\right],$$

where $s^*(\alpha) \to -\infty$, as $\alpha \to 0$, and $s^*(\alpha) \to \infty$, as $\alpha \to \pi$. The possibility of changing the lower limit $s^*(\alpha)$ for $-\infty$ will be discussed below, together with the estimation of the error thus made.

In the vicinities of the stationary points the slowly varying function $G(s)$ can be represented in the form (the complete expansion is given in [12]):

$$G(s) = b_0 + b_1 s, \tag{3.5}$$

$$b_0(\alpha) = \frac{G(\sqrt{\sigma}) + G(-\sqrt{\sigma})}{2}, \qquad b_1(\alpha) = \frac{G(\sqrt{\sigma}) - G(-\sqrt{\sigma})}{2\sqrt{\sigma}}. \tag{3.6}$$

The values of $d\psi/ds$ entering in $G(\pm\sqrt{\sigma})$ are determined by means of differentiating expression (3.2) with respect to the variable s

$$G(\sqrt{\sigma}) = f(\psi_2)\sqrt{\frac{-2\sqrt{\sigma(\alpha)}}{S''_{\psi\psi}(\psi_2,\ \alpha)}}, \qquad G(-\sqrt{\sigma}) = f(\psi_1)\sqrt{\frac{2\sqrt{\sigma(\alpha)}}{S''_{\psi\psi}(\psi_1,\ \alpha)}}. \tag{3.7}$$

Substituting expansion (3.5) in expression (3.4) we obtain

$$\mu(r,\ \alpha) = J_1(r,\ \alpha) + J_2(r,\ \alpha). \tag{3.8}$$

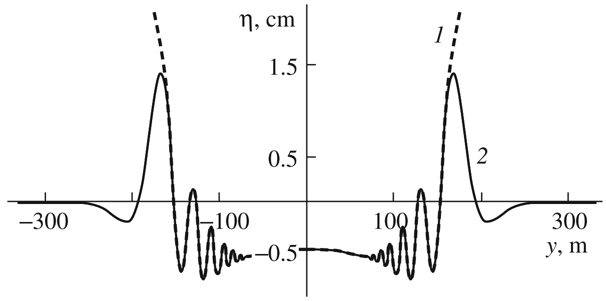

Fig. 2. Cross section of the elevation field: (*1*) stationary phase and (*2*) uniform asymptotics).

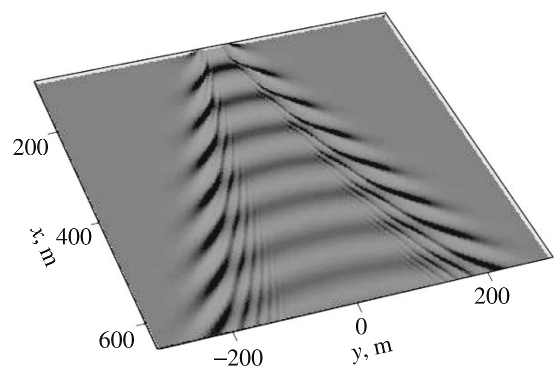

Fig. 3. Elevation field on the heavy fluid surface (uniform asymptotics).

Here, the first term can be expressed in terms of the Airy function $\mathrm{Ai}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \cos(xt - t^3/3)\,dt$ and the second term in terms of its derivative [12–14]

$$J_1(r,\ \alpha) = \frac{2\pi}{r^{1/3}} b_0(\alpha)\cos(ra_0(\alpha))\mathrm{Ai}(\sigma(\alpha)r^{2/3}),$$
$$J_2(r,\ \alpha) = \frac{2\pi}{r^{1/3}} b_1(\alpha)\sin(ra_0(\alpha))\mathrm{Ai}'(\sigma(\alpha)r^{2/3}), \tag{3.9}$$

where $b_0(\alpha)$ and $b_1(\alpha)$ are determined in Eq. (3.6) and $\sigma(\alpha)$ and $a_0(\alpha)$ in Eq. (3.3).

The uniform asymptotics (3.8) represent even function relative to $\alpha = \pi/2$. The wave pattern is also symmetric about the Oy axis, which is physically impossible, since far upstream disturbances are absent. This is due to the change of the lower integration limit $s^*(\alpha)$ in Eq. (3.4) for $-\infty$, which corresponds to the change of the lower integration limit $\pi/2 + \alpha$ for $\pi/2$ in Eq. (2.2). Then the integrals obtained can be expressed in terms of the Airy function and its derivative. At $0 < \alpha < \pi/2$ this change gives only an addition of the order of $O(1/r^2)$, since the stationary points belong to the interval $(s^*(\alpha),\ \infty)$. At $\pi/2 < \alpha < \pi$ this change cannot be made but it can be shown that the original integral (2.2) is of the order of $O(1/r^2)$ (one can convince oneself in it doubly integrating Eq. (2.2) by parts).

As a result, in the former case the main contribution into the asymptotics of integral (2.2) is made by the stationary points (the order of the integral is $O(1/r^{1/3})$), while in the latter case the contribution in the integral is made only by the end point of the integration interval $s^*(\alpha)$ (the order of the integral is $O(1/r^2)$). Thus, the uniform asymptotics (3.8) work at large values of r and $0 < \alpha < \pi/2$, while at $\pi/2 < \alpha < \pi$ the wave field decreases as $1/r^2$. These asymptotics are regular on the wedge boundary at $\alpha = \alpha^*$, where $S'(\psi) = 0$ and $S''(\psi) = 0$, and in this case $b_0(\alpha^*) = G(0)$ and $b_1(\alpha^*) = G'(0)$.

To determine $G(0)$ one must convince oneself that the function $d\psi/ds$ is regular at $s = 0$ and for determining $G'(0)$ $d^2\psi/ds^2$ must be regular at zero. Then differentiating expression (2.4) three and four times with respect to the variable ψ we obtain

$$\frac{d\psi}{ds} = \left(-\frac{2}{S'''(\psi^*)}\right)^{1/3}, \quad s = 0,$$

$$\frac{d^2\psi}{ds^2} = -\frac{S^{\mathrm{IV}}(\psi^*)}{\sigma S'''(\psi^*)}\left(-\frac{2}{S'''(\psi^*)}\right)^{2/3}, \quad s = 0.$$

At large values of r and at α not too close to α^* the uniform asymptotics go over into the nonuniform asymptotics (2.3). One can convince oneself in it taking in Eq. (3.8) the asymptotics of the Airy function and its derivative for large positive values of the argument ($x \to \infty$) instead of the functions themselves [13, 14]:

$$\mathrm{Ai}(x) \sim x^{-1/4}\cos T/\sqrt{\pi}, \quad \mathrm{Ai}'(x) \sim x^{1/2}\sin T/\sqrt{\pi}, \quad T = 2x^{3/2}/3 - \pi/4.$$

Then both terms in Eq. (3.8) are of the order of $O(1/\sqrt{r})$. In the immediate vicinity of the wedge boundary only the first term (of the order of $O(1/r^{1/3})$ may be retained in Eq. (3.8), while outside the wedge both terms, as well as their asymptotics, are exponentially small.

In Fig. 2 a cross section of the elevation wave field ($x = 500$ m and $Q = 10^3$ m^3/s) calculated by formulas (2.3) and (3.8) is presented. In Fig. 3 the three-dimensional elevation wave pattern on the heavy fluid surface is plotted; it is calculated by formula (3.8), that is, it represents the uniform asymptotics of the solution.

Summary. The asymptotic solutions constructed are uniform and make it possible to describe the far fields of the surface disturbances produced by localized sources both inside and outside the Kelvin wave wedge. The asymptotics of the far fields of the surface wave disturbances make it possible to effectively calculate the main parameters of the wave fields and, moreover, to qualitatively analyze the solutions obtained. Thus, wide possibilities for an analysis of the general wave patterns are opened, which is important for the adequate formulation of the mathematical models of wave dynamics and for making express estimations during full-scale measurements of the wave fields in the sea medium.

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