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Internal Waves Excited by a Moving Source in a Medium of Variable Buoyancy

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Abstract—The problem of the far field of internal gravity waves generated by an oscillating point perturbation source moving in a vertically infinite layer of a stratified medium of variable buoyancy is considered. The analytical solution of the problem is obtained by two ways for a model quadratic buoyancy frequency distribution. In the first case the solution is expressed in terms of the eigenfunctions of the vertical spectral problem and the Hermite polynomials. In the second case the solution in the form of the Green's characteristic function is represented in terms of the functions of parabolic cylinder. The analytical solutions obtained make it possible to describe the amplitude-phase characteristics of the far fields of internal gravity waves in a stratified medium with variable Brunt-Väisälä frequency.

Key words: stratified medium, internal gravity waves, buoyancy frequency, far fields.

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The asymptotic methods of investigation of the analytical models of wave generation [1–5] are widely used in modern scientific investigations in analyzing the dynamics of internal gravity waves (IGW) in natural stratified media (ocean and Earth's atmosphere). In these media the internal gravity wave fields are two-dimensional and in many cases they are even three-dimensional; therefore, in the computational aspect the analysis of two- and three-dimensional unsteady wave motions is a fairly complex problem. Numerical models do not make it possible effectively to calculate the particular physical problems of wave dynamics of the ocean and Earth's atmosphere with regard to their actual variability. They are oriented to the solution of fairly general problems, require high computational powers, and do not always take into account the physical specifics of the solved problems. This restricts significantly their practical applicability. In addition, the use of powerful numerical algorithms requires verification and comparison with the solutions of model problems [6–8].

In the linear approximation the available approaches to description of the wave patterns of the excited fields of internal gravity waves are based on the representation of the wave fields by the Fourier integrals investigated using the asymptotic methods [4, 5, 9]. The aim of the present work is to study the far fields of internal gravity waves generated by an oscillating perturbation source moving in a vertically-infinite stratified medium of variable buoyancy.

1. FORMULATION OF THE PROBLEM

We will consider the problem of the far fields of internal gravity waves generated in the motion of a point oscillating source of perturbations of the power $Q = q\exp(i\omega t)$, where $q = \mathrm{const}$, in a vertically-infinite inviscid stratified medium. The source moves at a constant velocity V in the horizontal direction along the x axis, the z axis is directed upward, and the depth of position of the source is $-z_0$.

We will consider the steady-state wave oscillation regime. In the moving coordinate system, in the Boussinesq approximation we have the following equation, for example, for a vertical displacement of isopycnics $\eta(x, y, z)$ (equal-density lines with the same harmonic time dependence) [9]:

$$\left(i\omega + V\frac{\partial}{\partial x}\right)^2 \Delta\eta + N^2(z)\Delta_2\eta = Q\left(i\omega + V\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0},$$

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$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, \quad N^2(z) = -\frac{g}{\rho_0(z)}\frac{d\rho_0(z)}{dz}, \tag{1.1}$$

where $N^2(z)$ is the square of the Brunt-Väisälä frequency (buoyancy frequency), $\rho_0(z)$ is the unperturbed density of the medium as a function of the depth, g is the gravity acceleration, and $\delta(x)$ is the Dirac delta-function.

In what follows, we will use the model buoyancy frequency distribution in the form $N^2(z) = N_0^2 - 4\chi^2 z^2$, which is widely used in oceanologic calculations in studying the internal gravity wave dynamics in the presence of a jump in density of the sea medium. A remarkable feature of the World Ocean is the presence of the thermocline, i.e., the region of rapid temperature variation and, at the same time, the region of high stability of the buoyancy frequency. In the model representation the depth dependence of the Brunt-Väisälä frequency can differ from the empirical dependences. The quadratic buoyancy frequency distribution makes it possible to solve the problem analytically, whereas the use of the empirical dependences requires to employ only the numerical methods. However, as shown in numerous investigations, the basic qualitative results on the description of the internal gravity wave dynamics depend, as a rule, not on the specific form of the buoyancy frequency approximation but on the presence of the maximum of $N^2(z)$ in the layer of jump in the density of oceanic water [2, 3, 6, 10].

We will use the boundary condition in the form:

$$\eta \to 0, \quad \text{as} \quad z \to \pm\infty. \tag{1.2}$$

In the dimensionless coordinates and variables $x^* = x/L,\ y^* = y/L,\ z^* = z/L,\ \eta^* = \eta N_0 L^2/q$, $\omega^* = \omega/N_0$, $t^* = tN_0$, $L = N_0/2\chi$, $c = N_0^2/2\chi$, and $M = V/c$ the Eq. (1.1) can be written as follows (in what follows, we will omit the asterisk):

$$\left(i\omega + M\frac{\partial}{\partial x}\right)^2 \Delta\eta + (1 - z^2)\Delta_2\eta = \left(i\omega + M\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0}. \tag{1.3}$$

We will seek the solution of the problem (1.2)–(1.3) in the form of the Fourier integrals:

$$\eta(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} \varphi(\mu, \nu, z)\exp(-i(\mu x + \nu y))\, d\mu. \tag{1.4}$$

Then, in order to determine the function $\varphi(\mu, \nu, z)$ it is necessary to solve the following boundary-value problem:

$$\frac{\partial^2\varphi}{\partial z^2} + k^2((1 - z^2)(\omega - \mu M)^{-2} - 1)\varphi = \frac{-i}{(\omega - \mu M)}\frac{\partial\delta(z - z_0)}{\partial z_0},$$
$$\varphi(\mu, \nu, z) \equiv \varphi(k, z) \to 0, \quad \text{as} \quad z \to \pm\infty, \quad k^2 = \mu^2 + \nu^2. \tag{1.5}$$

We introduce the notation $p = \omega - \mu M$ and will solve Eq. (1.5) with the right-hand side $i\delta(z - z_0)/p$. We will seek the solution $\varphi(k, z)$ of the problem (1.5) in the form of the following series in the eigenfunctions of the corresponding homogeneous problem

$$\varphi(k, z) = \sum_{n=0}^{\infty} A_n(k)\varphi_n(k, z),$$

$$\frac{\partial^2\varphi_n(k, z)}{\partial z^2} + k^2\left((1 - z^2)p_n^{-2}(k) - 1\right)\varphi_n(k, z) = 0, \quad \varphi_n(k, -\infty) = \varphi_n(k, \infty) = 0. \tag{1.6}$$

2. CONSTRUCTION OF THE ANALYTICAL SOLUTIONS

The eigennumbers $p_n(k)$ and the orthonormal eigenfunctions $\varphi_n(k,z)$, orthogonal with the weight $1-z^2$ on an infinite interval, have the form [4, 5]:

$$p_n(k) = \left(\sqrt{(2n+1)^2+4k^2} - (2n+1)\right)/2k,$$
$$\varphi_n(k,z) = B_n(k) H_n\left(z/\alpha_n(k)\right)\exp\left(-z^2/2\alpha_n^2(k)\right),$$
$$\alpha_n(k) = \sqrt{p_n(k)/k}, \quad B_n(k) = k^{1/4}\left(\left(1+p_n^2(k)\right)\sqrt{\pi p_n(k)}n!2^{n-1}\right)^{-1/2}, \tag{2.1}$$

where $H_n(z)$ are the Hermite polynomials [11]. Substituting (2.1) in (1.5) and taking (1.6) into account, we can obtain

$$\varphi(k,z) = \sum_{n=0}^{\infty} \frac{ip\, p_n(k)\varphi_n(k,z)\varphi_n(k,z_0)}{k^2\left(p^2 - p_n^2(k)\right)}. \tag{2.2}$$

We can also obtain the solution in the form (2.2) using another method by constructing the Green's characteristic function of the problem (1.5) in the form:

$$\varphi(p,k,z) = \frac{iF_-(p,k,z_-)F_+(p,k,z_+)}{pW(p,k)},$$
$$F_\mp(p,k,z) = D_\lambda\left(\mp z\sqrt{2k/p}\right),$$
$$z_- = \min(z,z_0), \quad z_+ = \max(z,z_0), \quad \lambda = \left(k - |p| - kp^2\right)/2|p|. \tag{2.3}$$

The function of parabolic cylinder $D_\lambda(\tau)$ must satisfy the equation

$$\frac{\partial^2 D_\lambda(\tau)}{\partial\tau^2} + \left(\lambda + \frac{1}{2} - \frac{\tau^2}{4}\right) D_\lambda(\tau) = 0,$$

where $W(p,k) = -2\sqrt{\pi k}/\Gamma(-\lambda)\sqrt{p}$ is the Wronskian of the functions $F_-(p,k,z)$ and $F_+(p,k,z)$ and $\Gamma(\lambda)$ is the gamma-function [11–13].

Then, using the formula of complement for the gamma-function, we have: $1/\Gamma(-\lambda) = -\pi\Gamma(\lambda)\sin(\pi\lambda)/\pi$. Applying to (2.3) the Mittag-Leffler theorem on the expansion of a meromorphic function into partial fractions in poles of the denominator at $\lambda = n,\ n = 0,\ 1,\ 2, \ldots$, we can obtain the dispersion relation (2.1). Taking into account the relation $D_n(\sqrt{2}z) = 2^{-n/2}H_n(z)\exp(-z^2/2)$, we can also obtain the expression (2.2) [12, 13].

In integrating in the formula (1.4) with respect to the variable μ, it is necessary to take into account the poles of the integrand $\varphi(k,z)$ in (2.2) which can be determined from the equation

$$(\mu M - \omega)^2 = p_n^2(k). \tag{2.4}$$

In the general case Eq. (2.4) has from two to four real roots [9]. We will consider the case $M > 1$ and $\omega < 1$ in which (2.4) has two roots $\mu_{n1}(\nu)$ and $\mu_{n2}(\nu)$ for any real ν.

The first set of the roots $\mu_{n1}(\nu)$ must satisfy the equation $\mu M - \omega = p_n(k)$, the roots of this set must satisfy the inequalities $\omega/M < \mu_{n1}(\nu) < (1+\omega)/M,\ n = 0,\ 1,\ 2, \ldots$ The second set of the roots is determined by the equation $\omega - \mu M = -p_n(k)$ for whose roots the inequalities $(\omega - 1)/M < \mu_{n2}(\nu) < \omega/M$ hold.

Using the perturbation method, we can demonstrate that the contour of integration with respect to the variable μ passes above the poles. When $x < 0$, then closing the contour of integration upward, we obtain that the field is exponentially small: there are no real poles. When $x > 0$, then closing the contour of integration downward, for an individual wave mode $\eta_{ni}(x,y,z,t)$ $(i = 1,2)$, taking the harmonic time dependence into account, we obtain the expression

$$\eta_{ni}(x,y,z,t) = \frac{1}{2\pi}\int_0^\infty T_{ni}(\mu,\nu,z)\cos(\mu x - \omega t)\cos(\nu y)\, d\nu,$$

$$T_{ni}(\mu,\nu,z)=\frac{p_n^2(k)\varphi_n(k,z)\varphi_n(k,z_0)}{k^2\left(M\mp p_n'(k)\mu/k\right)}, \tag{2.5}$$

where the sign "−" must be taken in the denominator at $i=1$ (the value $\mu=\mu_{n1}(\nu)$) and the sign "+" must be taken in the denominator at $i=2$ (the value $\mu=\mu_{n2}(\nu)$).

3. PHASE AND WAVE PATTERNS OF THE EXCITED FIELDS

In what follows, in all the numerical calculations we will use the values $M=1.7$ and $\omega=0.6$. In Fig. 1 we have plotted three dispersion curves of the first set $\mu_{01}(\nu)$, $\mu_{11}(\nu)$, and $\mu_{21}(\nu)$ (curves *1–3*) and three dispersion curves of the second set $\mu_{02}(\nu)$, $\mu_{12}(\nu)$, and $\mu_{22}(\nu)$ (curves *4–6*). In Fig. 2 we have reproduced the vertical displacements of isopycnics of the zeroth mode $\eta_{01}(x,y,z,t)$ of the first set for $z_0=0.3$, $z=-0.2$, and $t=8$. In Fig. 3 we have reproduced the vertical displacements of isopycnics of the zeroth mode $\eta_{02}(x,y,z,t)$ of the second set for the same values of z_0, z, and t. In Fig. 4 we have reproduced the sum $\eta_{01}(x,y,z,t)+\eta_{02}(x,y,z,t)$.

In the approximation of the steady-state phase method the integrals (2.5) take the form ($i=1$):

$$\eta_{n1}(x,y,z,t)=B_{n-}+B_{n+},$$

$$B_{n\pm}=\frac{T_{n1}(\mu_{n1}(\nu_\pm),\nu_\pm,z)}{\sqrt{2\pi x(\pm b_n(\nu_\pm))}}\cos\left(-i(\mu_{n1}(\nu_\pm)x-\nu_\pm y\pm\pi/4)\right),\quad b_n(\nu)=\frac{\partial^2\mu_{n1}(\nu)}{\partial\nu^2}, \tag{3.1}$$

where $\nu_\pm$ are the roots of the equation $\dfrac{\partial\mu_{n1}(\nu)}{\partial\nu}=y/x$. The expression obtained is applicable inside the wave wedge whose each half-angle θ can be determined from the relation $\theta=\arctan(\mu_{n1}(\nu_n^*))$,

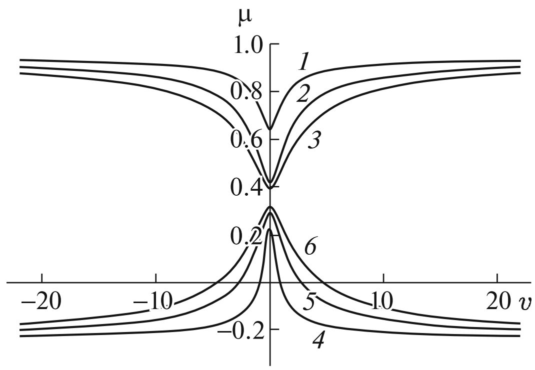

Fig. 1. Dispersion curves of the first three modes of $\mu_{n1}(\nu)$ (curves *1–3*) and $\mu_{n2}(\nu)$ (curves *4–6*).

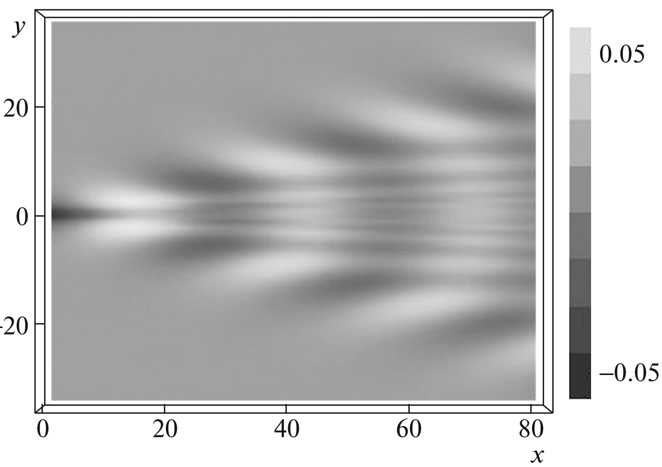

Fig. 2. Zeroth elevation mode η_{02}.

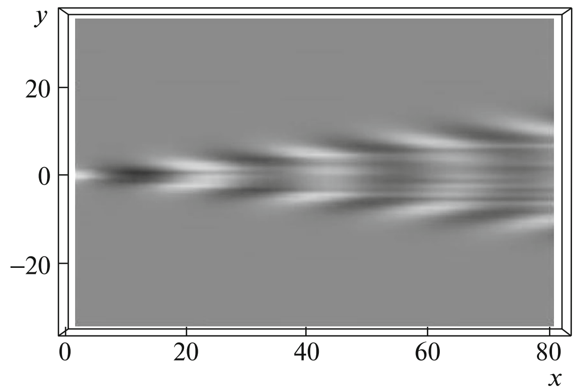

Fig. 3. First elevation mode η_{12}.

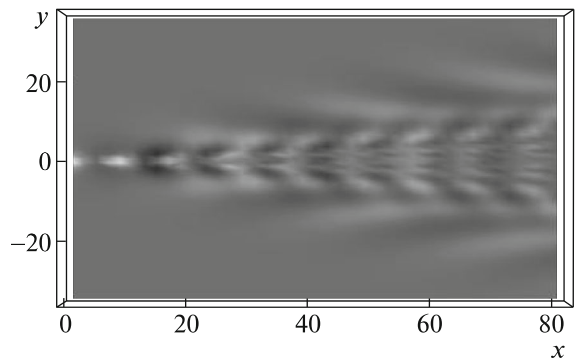

Fig. 4. Total elevation field $\eta_{02} + \eta_{12}$.

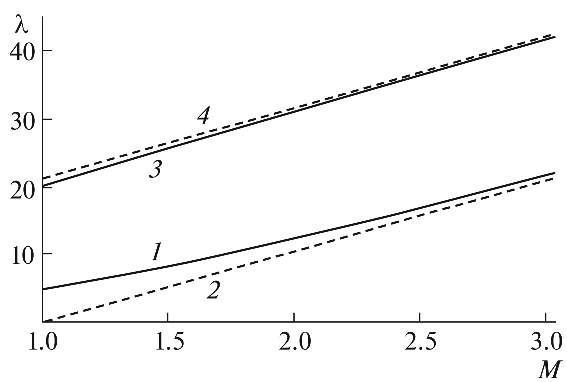

Fig. 5. Wavelength λ as a function of the number M for the zeroth modes: curves *1* and *2* correspond to η_{01} and curves *3* and *4* to η_{02}.

where ν_n^* is the root of the equation $b_n(\nu_n^*) = 0$. Similar estimates can also be obtained for the integrals $\eta_{n2}(x, y, z, t)$. The approximation (3.1) (nonuniform asymptotics) is applicable only inside the wave wedge. The asymptotics which describe the wave fields far away from the perturbation source is applicable both in the neighborhood and far away from the wave wedge (uniform asymptotics) can be expressed in terms of the Airy function and its derivative [4, 5, 9].

We will now consider the dependence of the wavelength of the excited fields of internal gravity waves, for example, along the x axis (when $y = 0$), on the M number at a fixed value of ω and on the frequency ω at a fixed value of M. The parameter M varies when the thermocline width N_0/χ varies and the value

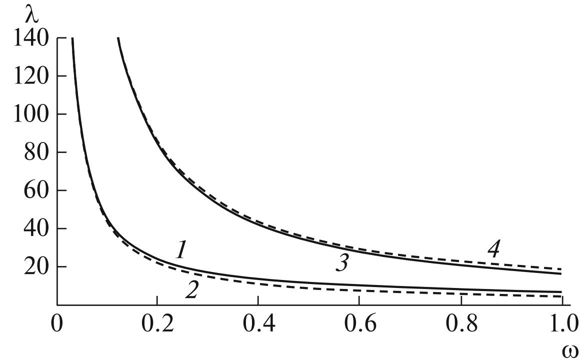

Fig. 6. Wavelength λ as a function of the frequency ω for the zeroth modes: curves *1* and *2* correspond to η_{01} and curves *3* and *4* to η_{02}.

of M increases with decrease in the thermocline width. The wavelength of the nth wave mode is equal to $\lambda = 2\pi/\mu_{ni}(0)$.

The numerical calculations showed that the contribution of the zeroth mode to the total wave field is up to 80%; therefore, in what follows, we will consider the case $n = 0$. In Fig. 5 we have plotted the graphs of the exact dependence of λ on M at a fixed value of $\omega = 0.6$ (curves *1* and *3*) and the approximate dependence (curves *2* and *4*) which has the form $\lambda \approx 2\pi(M-1)/\omega$ for η_{01} and $\lambda \approx 2\pi(M+1)/\omega$ for η_{02} (for small μ). From these graphs it follows that the wavelength of internal gravity waves increases with the M number.

In Fig. 6 we have plotted the graphs of the exact dependence of λ on ω at a fixed $M = 1.7$ (curves *1* and *3*) obtained by solving Eq. (2.4) numerically and the graphs of the approximate dependence (curves *2* and *4*). We can see that the length of the transverse internal gravity wave increases with ω along the x axis (when $\omega = 0$ there are no transverse waves).

SUMMARY

The problem of the far field of internal gravity waves generated by an oscillating point perturbation source moving in a vertically infinite layer of a stratified medium of variable buoyancy is solved.

The model quadratic distribution of the Brunt–Väisälä frequency is considered and it is shown that the analytical solution of the problem can be obtained by two ways. In the first case the solution is sought in the form of a series in the eigenfunctions of the corresponding homogeneous problem which can be represented in the form of the Hermite polynomials. In the second case the solution in the form of the Green's characteristic function, expressed in terms of the functions of parabolic cylinder, is obtained. It is shown that both methods of solving the problem are equivalent.

The dependences of the wave characteristics of the excited fields on the basic parameter of the model stratification used, namely, the thermocline width, and on the perturbation source oscillation frequency are investigated.

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