ISSN 0001-4370, Oceanology, 2018, Vol. 58, No. 6, pp. 796–801. © Pleiades Publishing, Inc., 2018.
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MARINE PHYSICS

Far Fields of Internal Gravity Waves from a Nonstationary Source

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Received June 26, 2017; in final form, February 12, 2018

Abstract—The problem of far fields of internal gravity waves from a nonstationary source moving in a stratified ocean of finite depth is considered. It is shown that the wave pattern of the generated far fields of internal waves for certain generation parameters is a system of hybrid wave perturbations that simultaneously have the properties of two wave types: annular (transverse) and wedge-shaped (longitudinal). The features of the phase structure and wave fronts of the generated fields are studied. Uniform asymptotics of the solutions describing far hybrid internal waves from a nonstationary source are constructed.

DOI: 10.1134/S0001437018060036

INTRODUCTION

An important mechanism for the excitation of fields of internal gravity waves in the ocean is their generation by perturbation sources of different physical nature: natural (a moving typhoon, flow over the ocean relief, and seamounts) and anthropogenic (marine engineering structures, collapse of a turbulent mixing zone, underwater explosions) [1, 6–12]. Fundamentally, the system of hydrodynamic equations describing wave perturbations is quite a complex mathematical problem, and the main results of solving problems on internal wave generation are represented in the most general integral form [1, 9, 11]. In this case, for their study, the obtained integral solutions require the development of asymptotic methods allowing for quantitative analysis and express estimates of the obtained solutions when measuring internal waves in the ocean. The existing approaches to describing wave patterns in a linear approximation represent the wave fields as Fourier integrals and analyze their asymptotics by the stationary phase method or by geometrical construction of the total envelopes of wave fronts using the kinematic theory of dispersive waves [5, 9–11]. Kinematic theory make it possible, among other things, to represent the phase surfaces of interval gravity wave fields analytically. The general problems involved in constructing phase patterns of dispersive waves generated by moving localized sources are described in [5]. The aim of this work is to solve the mathematically more complicated problem of constructing asymptotics that describe the features of both the phase and amplitude structures of far fields of internal gravity waves excited by an oscillating perturbation source moving in a stratified ocean of finite depth.

FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF SOLUTIONS

We consider the problem on far fields of internal gravity waves generated during the movement of a point source of perturbations with power Q in a stratified layer of a medium with depth H. It is assumed that the source's power depends harmonically on time $Q = q\exp(i\omega t)$. The source moves at velocity V in the negative direction along the x axis, the z axis is directed upward, and the depth of source occurrence is $-z_0$; we consider a steady regime of wave motions. The equation for the vertical prominence of isopycnals $\eta(x, y, z)$ (lines of equal density with the same harmonic time dependence) in a coordinate system moving simultaneously with the source has the following form in the linear formulation and with respect to the Boussinesq approximation [1, 2]:

$$\left(i\omega + V\frac{\partial}{\partial x}\right)^2 \Delta\eta + N^2(z)\Delta_2\eta = Q\left(i\omega + V\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0}, \quad (1.1)$$

$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $N^2(z) = -\dfrac{g}{\rho_0(z)}\dfrac{d\rho_0(z)}{dz}$ is the Brunt–Väisälä frequency, which further is assumed to be constant ($\rho_0(z)$ is the unperturbed density of the medium) and $\delta(x)$ is the Dirac delta function. On the one hand, analytical ocean stratification models ($N(z)$ = const, linear or other model distribution of the Brunt–Väisälä frequency) considerably facilitate mathematical solution of the problems; on the other, they raise questions on the adequacy and physical validity of these model representations. The used approximation of constancy of the Brunt–Väisälä frequency is one of the most popular in studying the dynamics of internal waves in the ocean. For certain regions of the World Ocean (the Arctic Basin), this approximation ($N(z)$ = const and rigid-lid) well describes the actual hydrology and is among the basic ones in real oceanological and hydrophysical calculations [1, 3, 4, 8].

The function $\eta(x, y, z)$ is associated with the vertical component of the internal wave velocity $w(x, y, z)$ by the relationship $w(x,y,z) = \left(i\omega + V\dfrac{\partial}{\partial x}\right)\eta(x,y,z)$ [2]. The rigid-lid condition is used as the boundary conditions:

$$\eta = 0 \text{ for } z = 0, -H. \tag{1.2}$$

In dimensionless coordinates, $x^* = x\pi/H$, $y^* = y\pi/H$, $z^* = z\pi/H$, $\eta^* = \eta H^2 V/q\pi^2$, $\omega^* = \omega/N$, $t^* = tN$, Eq. (1.1) and boundary conditions (1.2) are rewritten as follows (the superscript "*" is omitted hereinafter):

$$\begin{gathered}\left(i\omega + M\frac{\partial}{\partial x}\right)^2 \Delta\eta + \Delta_2\eta \\ = \left(i\omega + M\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0}, \\ \eta = 0 \text{ for } z = 0, -\pi,\end{gathered} \tag{1.3}$$

where $c = NH/\pi$ is the maximum value of the group velocity of the first mode of internal gravity waves in a stratified layer of the medium with depth H [1, 2], $M = V/c$. The case of $M > 1$ was studied in [2], and it was shown that far from the oscillating perturbation source, the excited fields are a system of longitudinal (wedge-shaped) waves that are confined within the corresponding wave fronts. This work examines the case of $M < 1$. The solution to problem (1.3) is sought in the form of the Fourier integral

$$\eta(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} \varphi(\mu,\nu,z) \times \exp(-i(\mu x + \nu y))d\mu.$$

To determine the function $\varphi(\mu,\nu,z)$, we have the following boundary-value problem (μ and ν are the horizontal components of the wave vector $\mathbf{k}$: $k^2 = \mu^2 + \nu^2$):

$$\begin{gathered}\frac{\partial^2\varphi(\mu,\nu,z)}{\partial z^2} + k^2\left(\frac{1}{(\omega - \mu M)^2} - 1\right)\varphi(\mu,\nu,z) \\ = \frac{i}{(\omega - \mu M)}\frac{\partial\delta(z - z_0)}{\partial z_0}, \\ \varphi = 0 \text{ for } z = 0, -\pi.\end{gathered} \tag{1.4}$$

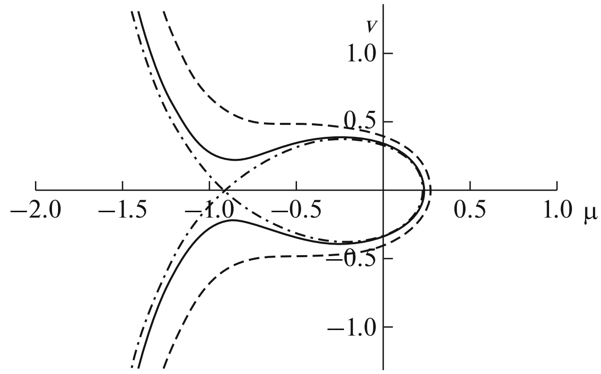

Fig. 1. Left branch of dispersion curve.

We represent the solution to problem (1.4) as the sum of the vertical modes:

$$\varphi(\mu,\nu,z) = \sum_{n=1}^{\infty}\varphi_n(\mu,\nu,z) = \sum_{n=1}^{\infty} B_n(\mu,\nu)\cos nz_0 \sin nz,$$

$$B_n(\mu,\nu) = \frac{2ni}{\pi(\omega - \mu M)}\frac{1}{k^2\left((\omega - \mu M)^{-2} - 1\right) - n^2},$$

i.e., as a sequence by eigenfunctions of uniform boundary-value problem (1.4). Hence, the solution to problem (1.3) has the form

$$\begin{gathered}\eta(x,y,z) = \sum_{n=1}^{\infty}\eta_n(x,y)\cos nz_0 \sin nz, \\ \eta_n(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} B_n(\mu,\nu)\exp(-i(\mu x + \nu y))d\mu.\end{gathered} \tag{1.5}$$

By equating the denominator to zero in $B_n(\mu,\nu)$, we can obtain a dispersion relation that relates the horizontal components μ and ν of the wave vector as

$$\nu_n(\mu) = \pm\left(\frac{n^2(\mu M - \omega)^2}{1 - (\mu M - \omega)^2} - \mu^2\right)^{1/2}, \quad n = 1, 2, .. \tag{1.6}$$

Then, we consider the case $\omega < 1$. Depending on the oscillation frequency of the perturbation source, the frequency interval $0 < \omega < 1$ is divided by two typical frequencies ω_n^1, ω_n^2 into three intervals: $0 < \omega < \omega_n^1$, $\omega_n^1 < \omega < \omega_n^2$, $\omega_n^2 < \omega < 1$, which determine the form of the dispersion curves, which consist of two branches. Figure 1 shows the left branch of dispersion curve (1.6), designated as $\nu_1^1(\mu)$ at $\omega = \omega_1^1$ (dash–dotted line), $\omega = \omega_1^2$ (dashed line), $\omega_1^1 < \omega < \omega_1^2$ (solid line) for the first mode $n = 1$ and $M = 0.4$. Figure 2 shows the right branch of the dispersion curve, denoted as $\nu_1^2(\mu)$ for the same parameters and notation.

In the first interval, the dispersion curve represents two open and one closed curve inside the dash-dotted separatrix in Fig. 1. In the second and third intervals, there are two open curves; in the second case, the left branch of the dispersion curve has two local extrema (solid line in Fig. 1), and in the third case, neither branch of the dispersion curves has any extrema.

The annular (transverse) waves correspond to the closed dispersion curves, and the wedge-shaped (longitudinal waves inside the Kelvin wedge), to the open curves. Here, in the third frequency interval, the half-angle of wave wedges is smaller than 90°, and in the second frequency interval, one of the two wedges has a half-angle greater than 90°. In the latter case, the generated waves have features of both annular (transverse) and wedge-shaped (longitudinal) waves, which will be referred to as hybrid.

We can determine the values of parameters ω and M characterizing the oscillation frequency and velocity of the perturbation source that generate annular waves for the mode with arbitrary number n. For this, it is expedient for $\nu = 0$ to find such a dependence $\omega_n^1(M)$ for which Eq. (1.6) has three roots. This dependence is determined by excluding variable μ from the system of equations:

$$F_n(\mu,0,\omega,M) = 0, \quad \frac{\partial F_n(\mu,0,\omega,M)}{\partial\mu} = 0, \quad \mu < 0,$$
$$F_n(\mu,\nu,\omega,M) \equiv (\mu M - \omega)^2 - (\mu^2+\nu^2)(\mu^2+\nu^2+n^2)^{-2}.$$

Hence, we obtain $\omega_n^1(M) = \left(1-(Mn)^{2/3}\right)^{3/2}$. If $M > 1/n$, no annular waves exist at any values ω for the nth mode. If $M < 1/n$, for $\omega > \omega_n^1(M)$, there are no annular waves, and for $\omega < \omega_n^1(M)$, they exist. This implies that, unlike the wedge-shaped waves, there are only a finite number of modes for which annular waves exist at any values of M. To calculate this number, the condition $1/(n+1) < M < 1/n$ should be fulfilled. Then, n is the maximum possible number of modes for which annular waves occur at the specified value of M. The $\omega_n^2(M)$ values are estimated from the system of equations

$$\frac{\partial\nu_n(\mu)}{\partial\mu} = 0, \quad \frac{\partial^2\nu_n(\mu)}{\partial\mu^2} = 0,$$

which is solved numerically. Figure 3 presents the dependences $\omega_1^1(M), \omega_1^2(M)$ for the first mode. If the oscillation frequency ω of the perturbation source lies below the curve of $\omega_1^1(M)$ for a certain fixed value of M, the wave field is a system of transverse (annular) and longitudinal (wedge-shaped) waves. If the value of ω lies above the curve of $\omega_1^2(M)$, the wave field is a system of longitudinal (wedge-shaped) waves only. If the value of ω is found between the both curves, the excited fields have the features of both longitudinal (wedge-shaped) and transverse (annular) waves.

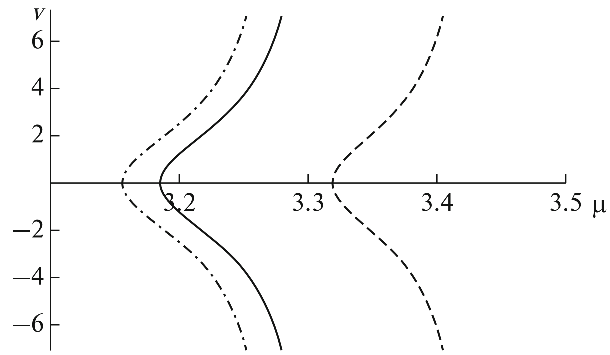

Fig. 2. Right branch of dispersion curve.

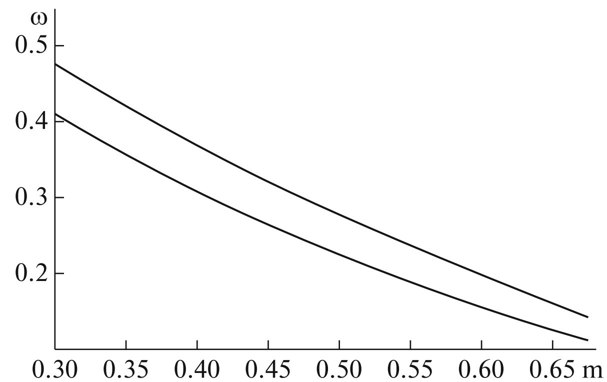

Fig. 3. Dependences of $\omega_1^1(M)$ (the lower curve) and $\omega_1^2(M)$ (upper curve).

Below, we consider hybrid internal waves ($M = 0.4$) and the first mode ($n = 1$, the subscript n is omitted). In this case, $\omega^1 = 0.309$, $\omega^2 = 0.370$, and all numerical calculations are presented for $\omega = 0.32$. Integration (1.5) using the variable ν is performed by the theory of residues. Upon integration with variable ν, the problem of passing around the poles on the real axis is solved by the perturbation method. The exact solution (with respect to the harmonic time dependence) is written as

$$\eta(x,y,t) = \eta(x,y)\exp(i\omega t) = J^1 + J^2,$$

$$J^1 = \frac{1}{2\pi}\int_{T_-}^{\alpha} A(\mu)\exp\left(-i(\mu x + \nu(\mu)|y| + \omega t)\right)d\mu,$$

$$J^2 = \frac{1}{2\pi}\int_{\beta}^{T_+} A(\mu)\exp\left(-i(\mu x - \nu(\mu)|y|) + \omega t\right)d\mu,$$

$$A(\mu) = \frac{\omega - \mu M}{\nu(\mu)(1-(\omega-\mu M)^2)}, \quad T_\pm = (\omega \pm 1)/M,$$

where α and β are the roots of the equation $\nu(\mu) = 0$.

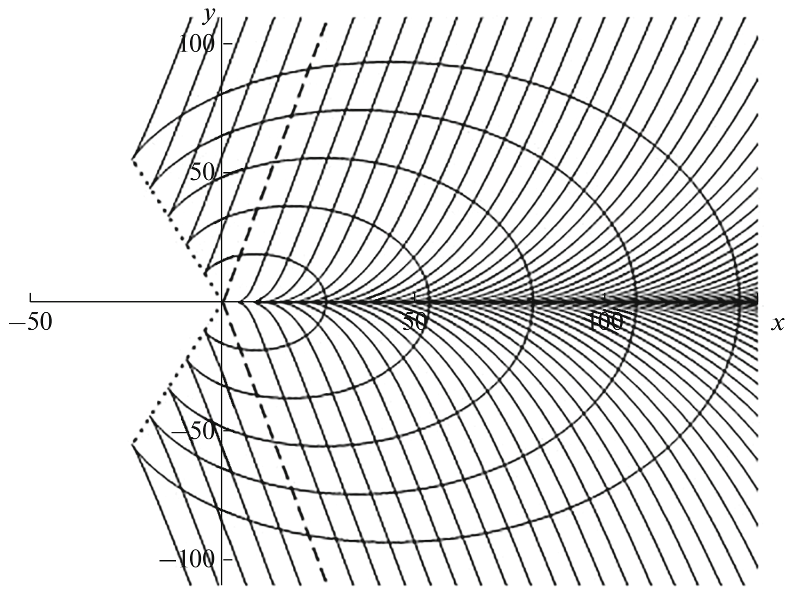

Fig. 4. Lines of equal phase.

ASYMTOTICS OF FAR WAVE FIELDS

J^2-type integrals describing the ordinary ship-generated internal gravity waves propagating far from perturbation sources with the dispersion dependence $\nu^2(\mu)$ were studied in [1, 2]. Here, we consider the term J^1 with the dispersion dependence $\nu^1(\mu)$ (the superscript *1* is omitted). We designate the phase $\Phi = \mu x + \nu(\mu)|y| + \omega t$ and, using the condition of phase stationarity $\frac{\partial\nu(\mu)}{\partial\mu} = x/|y|$, obtain a family of lines of the constant phase (with the parameter μ) $x = -\Phi\frac{\partial\nu(\mu)}{\partial\mu}\bigg/\left(\nu(\mu) - \mu\frac{\partial\nu(\mu)}{\partial\mu}\right)$, $|y| = \Phi\bigg/\left(\nu(\mu) - \mu\frac{\partial\nu(\mu)}{\partial\mu}\right)$. Since the asymptotics of the far wave fields are fully determined by the behavior of the stationary points of the phase function Φ that do not depend on ωt, this term is omitted hereinafter without loss of generality in order to simplify the calculations. To obtain the expression for the vertical prominence of the isopycnals $\eta(x, y, t)$, which is harmonically time-dependent, it suffices to multiply the asymptotic formulas obtained below by $\exp(i\omega t)$ and use the real part from the result. Figure 4 illustrates the equal phase lines for different values of $\Phi = 2\pi l$ (l is an integer). The aggregate of cuspidal points forms a wave front upstream of the flow (dashed line in Fig. 4). The corresponding half-angle of the wave wedge is 113°. The equation for the wave front is written as $x = \frac{\partial\nu(\mu)}{\partial\mu}|y|$, where $\mu = \mu_p$ is the root of the equation $\frac{\partial^2\nu(\mu)}{\partial\mu^2} = 0$. The dashed line in Fig. 4 is the crest of the wave with the phase $\Phi = 0$, and the equation of this line is written as $x = -\frac{\partial\nu(\mu)}{\partial\mu}|y|$, where $\mu = \mu_q$ is the root of the equation $\nu(\mu) = \mu\frac{\partial\nu(\mu)}{\partial\mu}$, which can be solved in explicit form: $\mu_q = (\omega - \omega^{1/3})/M$. As a result, we obtain the equation for the wave crest with a zero phase: $x = E|y|$, $E = (M^2(1-\omega^{2/3})^{-3} - 1)^{1/2}$. The annular and longitudinal crests to the left of the dashed line in Fig. 4 have the phase $\Phi > 0$, and the longitudinal crests to the right have the phase $\Phi < 0$. At infinity, i.e., at $\sqrt{x^2 + y^2} \gg 1$, the equation for the longitudinal wave crests has the form: $x = E|y| + 2\pi l M(1-\omega^{1/3})^{-1}$ (1 is an integer). Thus, these are the crests of a plane wave with a length of $\lambda = 2\pi\omega^{-1/3}(1-\omega^{2/3})^{1/2} \approx 6.7$. The length of annular waves in the direction of the x axis is $\lambda = 2\pi/T_- \approx 26.0$, i.e., greater by a factor of 4. The integral J^1 belongs to the class of integrals with two stationary points; when the stationary points are far from each other, the integral is determined by the stationary phase method. The stationary points merge at the wave front, and the stationary phase method is inapplicable in the vicinity of the front.

Following [1], we describe the procedure for constructing uniform asymptotics of the integral J^1. We introduce the notation $S(\mu,\rho) = \nu(\mu) - \rho\nu, \rho = x/|y|$ and substitute the variable

$$S(\mu,\rho) = a + \sigma s - s^3/3. \quad (2.1)$$

The right-hand side of (2.1) contains the simplest function with two merging (at $\sigma \to 0$) stationary points. We require that the point $s_1 = -\sqrt{\sigma}$ correspond to the stationary point $\mu_1(\rho)$, and the point $s_2 = \sqrt{\sigma}$, to the point $\mu_2(\rho)$. It then follows from (2.1) that

$$a(\rho) = 0.5(S(\mu_1(\rho),\rho)) + S(\mu_2(\rho),\rho)),$$

$$\sigma(\rho) = (0.75(S(\mu_2(\rho),\rho) - S(\mu_1(\rho),\rho)))^{2/3}.$$

After substitution of the variables (2.1), the integral J^1 has the form

$$J^1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(s)\exp\left(-i|y|(a + \sigma s - s^3/3)\right)ds, \quad (2.2)$$

$$G(s) = A(\mu(s))\frac{d\mu}{ds}.$$

The integration limits in (2.2) are replaced with infinite ones, and for large values $|y|$, the replacement error of the integration limits has an order of $O(1/|y|)$. The slowly changing amplitude $G(s)$ can be replaced with the linear function $L(s) = b_0 + b_1 s$, and it can be required that $L(-\sqrt{\sigma}) = G(-\sqrt{\sigma})$ for $s = -\sqrt{\sigma}$ and $L(\sqrt{\sigma}) = G(\sqrt{\sigma})$ for $s = \sqrt{\sigma}$. Then, $b_0(\rho) = 0.5(G(\sqrt{\sigma}) + G(-\sqrt{\sigma}))$, $b_1(\rho) = 0.5(G(\sqrt{\sigma}) + G(-\sqrt{\sigma}))/\sqrt{\sigma}$. The values of $\frac{d\mu}{ds}$ in the expression for $G(\pm\sqrt{\sigma})$ can be calculated by twice differentiating (2.1) using the variable s. Hence, we obtain

$$G(\sqrt{\sigma}) = A(\mu_2(\rho),\rho)\sqrt{\frac{-2\sqrt{\sigma(\rho)}}{\theta(\mu_2(\rho),\rho)}},$$

$$G(-\sqrt{\sigma}) = A(\mu_1(\rho),\rho)\sqrt{\frac{2\sqrt{\sigma(\rho)}}{\theta(\mu_1(\rho),\rho)}},$$

$$\theta(\mu(\rho),\rho) = \frac{\partial^2 S(\mu(\rho),\rho)}{\partial\mu^2}.$$

Substituting $L(s)$ in (2.2), we obtain the expression for uniform asymptotics $J^1(x,y)$ for large $|y|$:

$$J^1 = \frac{b_0(\rho)}{|y|^{1/3}}Ai(\tau)\exp(-iZ) - i\frac{b_1(\rho)}{|y|^{2/3}}Ai'(\tau)\exp(-iZ), \quad (2.3)$$

$$\tau = |y|^{2/3}\sigma(\rho), \quad Z = |y|a(\rho),$$

where $Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\tau t - t^3/3)dt$ is the Airy function and $Ai'(\tau)$ is the derivative Airy function [1]. Note that although the asymptotics of (2.3) is formally obtained for large $|y|$, this formula also works for $\sqrt{x^2 + y^2} \gg 1$.

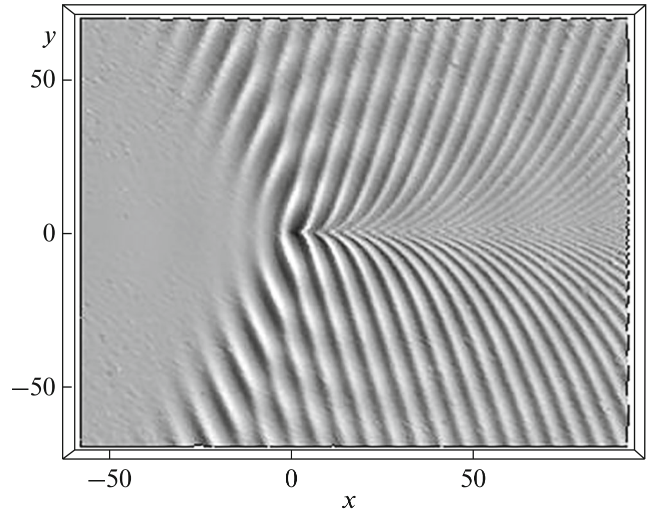

Fig. 5. Field of prominent internal gravity waves.

Figure 5 shows the wave pattern at $t = 10$, calculated using (2.3) in dimensionless coordinates (in the neighborhood of the origin, the integral $J^1(x,y)$ was estimated numerically). Using the asymptotics of the Airy function and its derivative far from the wave front, we can obtain nonuniform asymptotics for $J^1(x,y)$ consisting of two terms. The first term (corresponding to the root $\mu_1(\rho)$) describes wedge-shaped (longitudinal) waves, and the second one (corresponding to the root $\mu_2(\rho)$), annular (transverse) waves.

CONCLUSIONS

It is shown that the far fields of internal gravity waves from a nonstationary source moving in a stratified ocean of finite depth are a hybrid system of two wave types, annular (transverse) and wedge-shaped (longitudinal), under certain regimes of generation. The nonstationary amplitude of the perturbation source not only generates annular waves that propagate directly from the source, but also hybrid internal waves that propagate upward along the flow from the source. The quality pattern of the wave fields far from the nonstationary source is considerably complicated compared to the case of generation of internal waves by a moving stationary source when wave fronts of separate modes, starting from the first one, arrive consecutively at a fixed point [1, 12]. First, the annular waves may arrive first to a fixed observation point for certain generation parameters, in which case the number of incoming annular waves is always finite. Secondly, a mode front that differs from the first mode and has the largest half-angle of a Kelvin wedge may arrive at a fixed observation point, rather than the front of the first, second, and subsequent modes. The derived asymptotics make it possible to effectively calculate the main amplitude–phase characteristics of the excited far fields of gravity waves in nonstationary generation regimes and to qualitatively analyze the obtained solutions, which is important for correctly formulating mathematical models of real ocean wave dynamics.

ACKNOWLEDGMENTS

The study was carried out under state assignment topic no. AAAA-A17-117021310375-7.

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Translated by L. Mukhortova