Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Far surface gravity waves fields under unstable generation regimes

To cite this article: V V Bulatov and Yu V Vladimirov 2019 *J. Phys.: Conf. Ser.* **1392** 012005

Journal of Physics: Conference Series **1392** (2019) 012005 doi:10.1088/1742-6596/1392/1/012005

Far surface gravity waves fields under unstable generation regimes

V V Bulatov[1] **and Yu V Vladimirov**[2]

[1] Ishlinsky Institute for Problems in Mechanics RAS, 119526 Moscow, Russia
[2] Ishlinsky Institute for Problems in Mechanics RAS, 119526 Moscow, Russia

E-mail: internalwave@mail.ru, vladimyura@yandex.ru

Abstract. The problem of constructing uniform asymptotics of surface perturbations far fields from a localized harmonic source in the flow of a heavy homogeneous fluid of infinite depth is considered. It is shown that the wave pattern of excited long-range fields is, for certain generation parameters, a system of hybrid wave perturbations which simultaneously exhibit the properties of the following two types: annular-type (transverse) and wedge-type (longitudinal) waves. Specific features of the phase structure and wave fronts are studied. Uniform asymptotics of the solutions which describe the hybrid surface wave perturbations at a far distance from the harmonic source are constructed.

1. Introduction

The surface wave motions in the marine environment can either originate due to natural causes (wind waves, flow past underwater obstacles, bottom relief variations, density and flow fields) or be generated by the flow past natural obstacles (platforms, underwater pipelines, complex hydraulic facilities). The general system of hydrodynamic equations describing the surface perturbations is a rather complicated mathematical problem from the standpoint of proving the existence and uniqueness theorems for solutions in the corresponding function classes and from the computational standpoint. In the framework of the linear theory, the surface wave perturbations are analytically studied by integral representation methods and various asymptotic methods. The main results of solving the problems of generation of surface wave perturbations are represented in most general integral form, and to obtain the integral solutions, it is thus necessary to develop asymptotic methods for their investigation which admit a qualitative analysis and rapid estimations of the obtained solutions. Moreover, to analyze the data of the sea surface remote sensing, it is required to know the causes of various surface phenomena [1-7]. To obtain a detailed description of a wide class of physical phenomena related to the dynamics of surface perturbations in inhomogeneous and unsteady natural environments, it is necessary to have sufficiently developed mathematical models. The fact that the structure of the heavy sea surface is three-dimensional is also significant, and there are currently no possibilities for large-scale computational experimental modeling of three-dimensional ocean flows at large times with a sufficient accuracy . But in several cases, the initial qualitative concept of the considered class of wave phenomena can be obtained by using simpler asymptotic models and analytic methods for studying them [8-12]. In this connection, it is necessary to mention the classical hydrodynamic problems of constructing asymptotic solutions which describe the evolution of surface perturbations excited by sources of various nature in heavy homogeneous liquids. The model solutions permit further obtaining representations of surface wave fields with regard to variability and unsteadiness of real natural environments. So several results of asymptotic analysis of linear problems describing different regions of generation and propagation of surface perturbations also underlie the currently actively developing nonlinear theory of generation of ocean waves of extremely large amplitude, the so-called rogue waves (killer waves) [3,6] . The contemporary state of the art in the study of linear and nonlinear surface perturbations can be found in [7]. The problems of constructing uniform far field asymptotics of internal and surface perturbations from a moving source were considered in [2,5,13,14]. Therefore, it is interesting to consider more complicated regimes of surface wave generation due to the unsteady nature of the source of perturbations. The goal in this paper is to construct uniform far field asymptotics of surface perturbations generated in the flow of a heavy homogeneous liquid of infinite depth around a localized harmonic source of perturbations.

2. Main results and discussion

We consider the problem of homogeneous flow of an infinitely deep heavy liquid past a harmonic perturbation source of intensity $q = Q\,exp(i\omega t)$; the liquid has velocity V at a far distance from the source. The source is located at depth h with respect to the unperturbed free surface, i.e., at the point $(0,0,-h)$. To determine a physically realizable solution of the problem, it is necessary to replace the frequency ω by $\omega - i\varepsilon$ and then let ε tend to zero in the obtained solution for the free surface elevation. Further, let $\Omega(x,y,z,t)$ be the potential of velocity perturbations with respect to the velocity of the homogeneous flow. In the framework of the linear theory, to determine Ω, we have the problem [1,14]

$$\Delta\Omega(x,y,z,t) = Q\exp(i(\omega - i\varepsilon)t)\delta(x)\delta(y)\delta(z+h),\ z<0;\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2\Omega + g\frac{\partial\Omega}{\partial z} = 0,\ z = 0 \qquad (1)$$

where Δ is the three-dimensional Laplace operator and $\delta(x)$ is the Dirac delta function. The free surface elevation $H(x,y,t)$ is determined from the Cauchy-Lagrange integral

$$H(x,y,t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\Omega(x,y,z,t),\ z = 0\ . \qquad (2)$$

We seek the solution of equations (1)-(2) in the form

$$\Omega(x,y,z,t) = exp(i(\omega - i\varepsilon)t)\varphi(x,y,z),\ H(x,y,t) = exp(i(\omega - i\varepsilon)t)\eta(x,y).$$

In the dimensionless variables $x_* = gxV^{-2}$, $y_* = gyV^{-2}$, $z_* = gzV^{-2}$, $h_* = ghV^{-2}$, $\omega_* = V\omega/g$, $t_* = gt/V$, $\varepsilon_* = V\varepsilon/g$, $\varphi_* = V^2\varphi/Qg$, $\eta_* = V^3\eta/Qg$, we have the following problem for determining the functions $\varphi_*(x_*,y_*,z_*)$, $\eta_*(x_*,y_*)$ (the superscript «*» is further omitted):

$$\Delta\varphi(x,y,z) = \delta(x)\delta(y)\delta(z+h), z<0 \qquad \left(i\omega + \varepsilon + \frac{\partial}{\partial x}\right)^2\varphi + \frac{\partial\varphi}{\partial z} = 0,\ z = 0 \qquad (3)$$

$$\eta(x,y) = -\left(i\omega + \varepsilon + \frac{\partial}{\partial x}\right)\varphi, \quad z = 0 \qquad (4)$$

We represent the functions $\varphi(x,y,z)$, $\eta(x,y)$ as double Fourier integrals

$$\varphi(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} exp(-i\mu x)d\mu \int_{-\infty}^{\infty} exp(-i\nu y)f(\mu,\nu,z)\,d\nu \qquad (5)$$

$$\eta(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} exp(-i\mu x)d\mu \int_{-\infty}^{\infty} exp(-i\nu y)\Lambda(\mu,\nu)\, d\nu \tag{6}$$

Substituting expressions (5) in equation (3), we obtain the boundary-value problem

$$\frac{\partial^2 f(\mu,\nu,z)}{\partial z^2}-(\mu^2+\nu^2)f(\mu,\nu,z)=\delta(z+h), \quad z<0 \tag{7}$$

$$(i\omega+\varepsilon-i\mu)^2 f(\mu,\nu,z)+\frac{\partial f(\mu,\nu,z)}{\partial z}=0, \quad z=0$$

The solution of problem (7) in the domain $-h<z<0$ has the form

$$f(\mu,\nu,z)=-\frac{(\omega-\mu)^2 \mathrm{sh}(kz)+k\mathrm{ch}(kz)}{k\, exp(kh)((\varepsilon+i(\omega-\mu))^2+k)}, \quad k^2=\mu^2+\nu^2$$

The function $\Lambda(\mu,\nu)$ is determined from equation (4) as

$$\Lambda(\mu,\nu)=\frac{i(\omega-\mu)exp(-kh)}{(\varepsilon+i(\omega-\mu))^2+k} \tag{8}$$

The zeros of the denominator in (8) determine the dispersion relation $(\omega-\mu)^2=(\mu^2+\nu^2)^{1/2}$ which can be explicitly written as

$$\nu(\mu)=\pm((\omega-\mu)^4-\mu^2)^{1/2} \tag{9}$$

The set of frequencies $\omega>0$ is divided by two characteristic values $\omega_1=0.25$ and $\omega_2=\sqrt{6}/9$ into three intervals. For $\omega<\omega_1$, the dispersion curve determined from equation (9) consists of three branches: one closed and two unclosed. Then the wave pattern is the sum of two ship (longitudinal) waves with half-opening angle of the wave wedge less than $\pi/2$ and annular (transverse) waves around the source. For $\omega>\omega_2$, the dispersion curve consists of two unclosed branches without extrema. In this case, the wave pattern is the sum of two ship waves with half-opening angle of the wave wedge less than $\pi/2$. If $\omega_1<\omega<\omega_2$, then the dispersion curve consists of two unclosed curves one of which has two local extremes. One branch of the dispersion curve corresponds to usual ship waves with half-opening angle of the wave wedge less than $\pi/2$, and the second branch, to ship waves with half-opening angle of the wave wedge greater than $\pi/2$ (the wave front is directed upstream away from the source). This system of hybrid waves simultaneously has characteristic features of both annular (transverse) and ship (longitudinal) waves. We further consider this case ($\omega=0.255$). Figure 1 shows the dispersion curve branch (further denoted by $\nu_1(\mu)$) which describes the hybrid waves. Figure 2 shows the branch of dispersion curve (9) (further denoted by $\nu_2(\mu)$) which describes the ship waves. For $\varepsilon>0$ and $\mu<\omega$, we have $Im\,\nu_1(\mu)<0$, and for $\mu>\omega$, we have $Im\,\nu_2(\mu)>0$. Then, calculating the inner integral in equation (6) by closing the contour of integration in the variable ν for $y>0$ in the lower half-plane (the poles $\nu_1(\mu)$ and $-\nu_2(\mu)$), and for $y<0$, in the upper half-plane (the poles $-\nu_1(\mu)$ and $\nu_2(\mu)$), we obtain

$$\eta(x,y)=I_1(x,y)+I_2(x,y)$$

$$I_1(x,y)=\frac{1}{2\pi}\int_{-\infty}^{C}\frac{E}{\nu_1(\mu)}exp(-i(\mu x+\nu_1(\mu)|y|))d\mu\,,\ I_2(x,y)=\frac{1}{2\pi}\int_{D}^{\infty}\frac{E}{\nu_2(\mu)}exp(-i(\mu x-\nu_2(\mu)|y|))d\mu$$

where $E=(\omega-\mu)^3 exp(-(\omega-\mu)^2 h)$, C and D are the abscissas of the rightmost and leftmost points of the dispersion curves $\nu_1(\mu)$ and $\nu_2(\mu)$ in figures 1 and 2, respectively. Points A and G are points of inflection of the dispersion curves $\nu_1(\mu)$ and $\nu_2(\mu)$ in figures 1 and 2, respectively. The integrals $I_2(x,y)$ with the corresponding dispersion dependence $\nu_2(\mu)$, which describe the usual ship waves, were studied in detail, for example, in [2,5,13,14]. A more complicated and yet unstudied wave pattern of amplitude-phase characteristics of hybrid surface wave perturbations is described by the integrals $I_1(x,y)$. We denote the phase by $\Phi=\nu_1(\mu)|y|+\mu x$. Then, using the phase stationary condition in the form

$$\frac{d\nu_1(\mu)}{d\nu}=-\frac{x}{|y|} \tag{10}$$

we obtain the family of lines of constant phase with a parameter μ (the subscript «1» is omitted)

$$x=-\frac{\Phi\nu'(\mu)}{\nu(\mu)-\mu\nu'(\mu)},\qquad |y|=\frac{\Phi}{\nu(\mu)-\mu\nu'(\mu)} \tag{11}$$

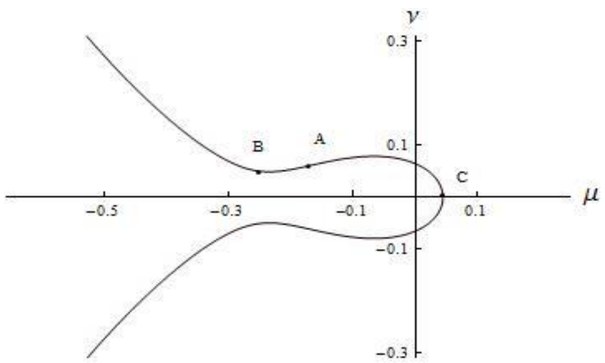

Figure 1. Dispersion curve $\nu_1(\mu)$

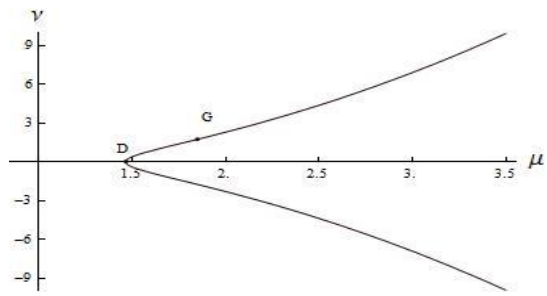

Figure 2. Dispersion curve $\nu_2(\mu)$

Figure 3 presents the lines of equal phase for different values of Φ with step 2π . The part of the dispersion curve from point C to the inflection point A corresponds to annular (transverse) waves, and the set of corner cusps forms the wave front moving upstream, which is depicted in figure 3 by a dotted line. To determine the wave front, it is necessary to supplement equation (11) with the condition $\frac{\partial x}{\partial\mu}\frac{\partial|y|}{\partial\Phi}-\frac{\partial x}{\partial\Phi}\frac{\partial|y|}{\partial\mu}=0$ or, which is the same, $\frac{d^2\nu(\mu)}{d\mu^2}=0$, where the abscissa μ_A of point A is a solution of the equation under study. Then the equation for determining the wave front becomes $x=-\nu'(\mu_A)|y|$ and the corresponding half-opening angle of the wave wedge is equal to $105^0 7'$. The part of the dispersion curve from point A to point B corresponds to the longitudinal crests of waves propagating from the front to infinity (depicted on the left in figure 3 by a dashed line). The dashed line in figure 3 corresponds to the crest of the wave with phase $\Phi=0$ and is described by the equation $x=-\nu'(\mu_B)|y|$ or $x=\sqrt{16\omega^2-1}|y|$, where μ_B is a root of the equation $\nu(\mu)=\mu\nu'(\mu)$

whose solution is $\mu_B = -\omega$, $\nu'(\mu_B) = -\sqrt{16\omega^2 - 1}$. The part of the dispersion curve to the left of point B corresponds to longitudinal crests of waves propagating from infinity to the origin (to the right of the dashed line in figure 3). In this figure, the phases Φ corresponding to the part of the dispersion curve to the right of point B are equal to $2\pi n, n = 1,2,3,4$, and to the left of point B, to $2\pi n, n = -11,-10,..,-1$. At infinity (for large values of x, $|y|$), the equations of crests of longitudinal waves have the form $x = \sqrt{16\omega^2 - 1}|y| - 2\pi k/\omega$, where k is integer, i.e., these crests are crests of a plane wave of length $\lambda = \pi\omega^{-2} \approx 24$. The wave length of annular (transverse waves) in the direction of the axis x is equal to $\lambda = 2\pi / C \approx 142$, i.e., is approximately six times greater than the length of wedge-shaped (longitudinal) waves.

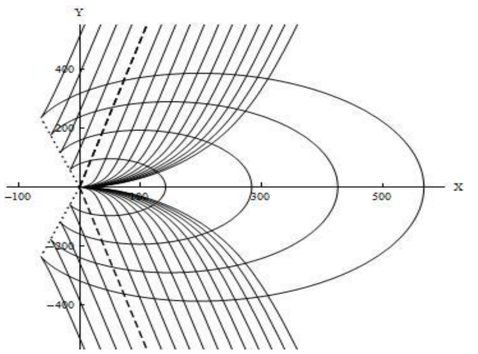

Figure 3. Phase structure of surface gravity waves

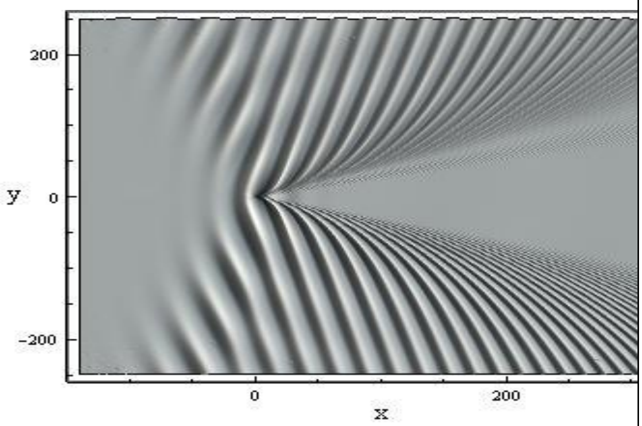

Figure 4. Surface gravity waves from moving unstable source

The integral $I_1(x,y)$ belongs to the class of integrals with two stationary points [15]. On the wave front (the dotted line in figure 3), the stationary points determined by equation. (10) merge. The uniform asymptotics $I_1(x,y)$ for large value of $|y|$ is constructed similarly [13-14] and has the form

$$I_1(x,y) = \frac{T^+(\rho)}{|y|^{1/3}} Ai(|y|^{2/3}\sigma(\rho))exp(-i|y|a(\rho)) - i\frac{T^-(\rho)}{|y|^{2/3}\sqrt{\sigma(\rho)}} Ai'(|y|^{2/3}\sigma(\rho))exp(-i|y|a(\rho))$$

$$T^{\pm}(\rho) = \frac{1}{2}\left(F(\mu_2)\sqrt{\frac{-2\sqrt{\sigma(\rho)}}{\theta(\mu_2,\rho)}} \pm F(\mu_1)\sqrt{\frac{2\sqrt{\sigma(\rho)}}{\theta(\mu_1,\rho)}} \right) \quad F(\mu) = (\omega-\mu)^3 exp(-h(\omega-\mu)^2)/\nu(\mu)$$

$$\sigma(\rho) = \left(\frac{3}{4}\left(S(\mu_2,\rho) - S(\mu_1,\rho)\right)\right)^{2/3}, \quad a(\rho) = \frac{1}{2}\left(S(\mu_2,\rho) + S(\mu_1,\rho)\right)$$

$$S(\nu,\rho) = \nu(\mu) - \rho\mu, \quad \rho = -x/|y|, \quad \theta(\nu,\rho) = \frac{\partial^2 S(\mu,\rho)}{\partial\mu^2}, \quad Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} cos\left(\tau u - u^3/3\right)du$$

where $\mu_1 = \mu_1(\rho)$ and $\mu_2 = \mu_2(\rho)$ are roots of the equation $\partial S(\mu,\rho)/\partial\mu = 0$, $|\mu_1| < |\mu_2|$, $Ai(\tau)$ is the Airy function, and $Ai'(\tau)$ is the derivative of the Airy function [15]. To determine the free surface elevation, it is necessary to multiply expression for $I_1(x,y)$ by $exp(i\omega t)$ and take the real part of the obtained result. Figure 4 presents the wave pattern of the free surface elevation for $t=10$ and $h=3$ which is calculated in dimensionless coordinates (near the origin, the integral $I_1(x,y)$ was calculated numerically). For the parameter values which are typical of the ocean conditions $Q=10^3 m^3/sec, V=3m/sec$, the elevation amplitudes are of the order of 0.3 meter. Using the asymptotics of the Airy function and its derivative at a far distance from the front, we can obtain a non-uniform asymptotics for $I_1(x,y)$ consisting of two terms. The first of these terms corresponding to the root μ_1 describes wedge-shaped (longitudinal) waves, and the second term corresponding to the root μ_2 describes annular (transverse) waves. Thus, we have shown that, in certain generation regimes, the far fields of surface perturbations from a non-stationary source localized in the flow of a heavy liquid of infinite depth form a hybrid system of waves of the following two types: annular (transverse) and wedge-shaped (longitudinal). The qualitative picture of wave fields at a far distance from a nonstationary source is significantly more complicated compared to the case of wave generation by a moving stationary source when the wave fronts come to a fixed observation point. The unsteadiness of the perturbation source amplitude results not only in the appearance of annular waves diverging on the liquid surface directly from the source but also in generation of hybrid surface perturbations propagating upstream from the source. The obtained asymptotics of surface wave perturbations far field allow one efficiently to calculate the basic amplitude-phase characteristics of wave fields and, in addition, qualitatively to analyze the obtained solutions, which is important in developing of well-posed mathematical models of wave dynamics of surface perturbations of the real natural environments. This opens wide opportunities for investigating the wave fields in general, which is also important for formulating correct statements of mathematical models of wave dynamics and for obtaining express evaluations in the surface field measurements in ocean. The research was carried out in the framework of the Federal target program No. AAAA-A17-117021310375-7.

References

[1] Lighthill M J 1978 *Waves in fluids* (Cambridge: Cambridge University Press).
[2] Bulatov V V and Vladimirov Yu V 2007 *Internal gravity waves: theory and application*s (Moscow: Nauka Publishers).
[3] Kharif C, Pelinovsky E and Slynyaev A 2009 *Rogue waves in the ocean* (Berlin–Heidelberg: Springer).
[4] Pedlosky J 2010 *Waves in the ocean and atmosphere: introduction to wave dynamics* (Berlin–Heidelberg: Springer).
[5] Bulatov V V and Vladimirov Yu V 2012 *Wave dynamics of stratified mediums* (Moscow: Nauka Publishers).
[6] Shurgalina E G and Pelinovsky E N 2015 *Dynamics of irregular wave ensembles in the coastal zone* (Nizhny Novgorod: Nizhny Novgorod State Technical University n.a. R.E.Alekseev Publishing).
[7] Mei C C, Stiassnie M and Yue D K-P 2017 *Theory and applications of ocean surface waves* (London: World Scientific Publishing).
[8] Lannes D 2005 *J.American Math. Society* **18** 605.
[9] Brown M G 2000 *Wave Motion* **32** 247.
[10] Zakharov V E , Dyachenko A I and Vasilyev O A 2002 *European J. Mech.. B Fluids* **21** 283.
[11] Ruban V P 2005 Phys.Lett.A **340** 194.
[12] Dobrokhotov S Yu, Grushin V V , Sergeev S A and Tirozzi 2016 *Russian J. Math. Physics* **23** 455.
[13] Bulatov V V and Vladimirov Yu V 2015 *Fluid Dynamics* **50** 741.
[14] Bulatov V V, Vladimirov Yu V and Vladimirov I Yu 2017 *Fluid Dynamics* **52** 617.
[15] Haberman R 2013 *Applied partila differential aequtions* (NJ: Pearson Educations).