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INTERNAL GRAVITY WAVES EXCITED BY A MOVING OSCILLATING SOURCE IN A STRATIFIED MEDIUM WITH VARIABLE BUOYANCY

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Abstract: A problem of a far field generated by internal gravity waves from an oscillating point source of perturbations that moves in a vertically infinite stratified medium with variable buoyancy. For a simulated quadratic distribution of buoyancy frequencies, analytical solutions of the main boundary-value problem are obtained, expressed through parabolic cylinder functions. Asymptotic solutions constructed in this paper make it possible to describe amplitude–phase characteristics of far fields generated by internal gravity waves in a stratified medium with a variable Brunt–Väisälä frequency.

Keywords: stratified medium, internal gravity waves, variable buoyancy frequency.

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The dynamics of internal gravity waves (IGWs) in natural stratified media (ocean, Earth's atmosphere, etc.) is often analyzed by means of asymptotic methods for studying the analytical models of wave generation [1–6]. In these media, IGW fields are two-dimensional and quite often three-dimensional, so the numerical investigation of two-dimensional and three-dimensional nonstationary wave motions is a very complex task. Numerical models make it impossible to effectively solve physical problems of wave dynamics of the ocean and atmosphere with allowance for their real variability, are aimed at solving general problems, require a high computational power, and not always account for the specificity of problems solved, which significantly limits the span of their use in practice. Moreover, the use of numerical algorithms requires verification and comparison with the solutions of model problems [4, 6]. In a linear approximation, the existing approaches to describing the wave pattern of excited IGW fields are based on representing wave fields using Fourier integrals and carrying out an asymptotic analysis of the solutions obtained [5, 7, 8].

The purpose of this paper is to study the far fields of IGWs, excited by an oscillating source of perturbations, which moves in an infinite vertical in a stratified medium with variable buoyancy.

1. FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF SOLUTIONS

We consider a problem of the far fields of IGWs, occurring in the motion of the point source of perturbations with a capacity $Q = q\,e^{i\omega t}$ ($q = \mathrm{const}$), which pulsates according to a harmonic law in a vertically infinite nonviscous stratified medium. The source moves with a constant velocity V in the direction of the x axis, the z axis is directed upward, and the depth of location of the source is z_0. Steady-state wave oscillations are under consideration. In a moving coordinate system, in the Boussinesq approximation, the following equation describes the vertical displacement of isopycnics $\eta(x, y, z)$ (equal-density lines with a harmonic dependence on time) [7]

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$$\Big(i\omega + V\,\frac{\partial}{\partial x}\Big)^2 \Delta\eta + N^2(z)\Delta_2\eta = Q\Big(i\omega + V\,\frac{\partial}{\partial x}\Big)\delta(x)\delta(y)\,\frac{\partial\delta(z-z_0)}{\partial z_0},$$

$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \qquad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad N^2(z) = -\frac{g}{\rho_0(z)}\,\frac{d\rho_0(z)}{dz}, \tag{1.1}$$

where $N(z)$ is the Brunt–Väisälä frequency (buoyancy frequency), $\rho_0(z)$ is the unperturbed density of the medium by depth, g is the acceleration of gravity, and $\delta(x)$ is the Dirac delta function. Next, the model distribution of the buoyancy frequency is used in the form $N^2(z) = N_0^2 - 4\chi^2 z^2$, which is widely used to study the IGW dynamics in the presence of a constant thermocline (density transition zone) [1, 3, 5, 8]. The special feature of the world's oceans and seas is the presence of a constant density transition zone, which a zone of rapid temperature change and simultaneously buoyancy frequency stability. The theoretical dependence of the Brunt–Väisälä frequency on depth may differ from empirical dependences, which also have a maximum $N^2(z)$ in the density transition zone of the water medium. The quadratic distribution of the buoyancy frequency makes it possible to solve this problem analytically, while the use of empirical dependences needs numerical methods. However, the main qualitative results of studying the IGW dynamics usually depend on the existence of the function maximum $N^2(z)$ in the density transition zone of the oceanic water rather than a particular analytical form of approximation of the buoyancy frequency [1, 3, 4, 9].

The boundary condition has the form

$$\eta \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{1.2}$$

In dimensionless coordinates and variables

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad \eta^* = \frac{\eta N_0 L^2}{q}, \quad \omega^* = \frac{\omega}{N_0},$$

$$t^* = tN_0, \quad L = \frac{N_0}{2\chi}, \quad c = \frac{N_0^2}{2\chi}, \quad M = \frac{V}{c}$$

Eq. (1.1) is written as follows (the superscript "$*$" is omitted below):

$$\Big(i\omega + M\,\frac{\partial}{\partial x}\Big)^2 \Delta\eta + (1-z^2)\Delta_2\eta = \Big(i\omega + M\,\frac{\partial}{\partial x}\Big)\,\delta(x)\,\delta(y)\,\frac{\partial\delta(z-z_0)}{\partial z_0}. \tag{1.3}$$

The solution of problem (1.2), (1.3) is determined in the form of the Fourier integrals

$$\eta(x,y,z) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} d\nu \int\limits_{-\infty}^{\infty} \varphi(\mu,\nu,z)\,\mathrm{e}^{-i(\mu x+\nu y)}\,d\mu. \tag{1.4}$$

Then, the function $\varphi(\mu,\nu,z)$ should be determined by solving the boundary-value problem

$$\frac{\partial^2\varphi}{\partial z^2} + k^2\Big(\frac{1-z^2}{(\omega-\mu M)^2} - 1\Big)\varphi = -\frac{i}{\omega-\mu M}\,\frac{\partial\,\delta(z-z_0)}{\partial z_0},$$

$$\varphi(\mu,\nu,z) \to 0, \quad z \to \pm\infty, \qquad k^2 = \mu^2 + \nu^2. \tag{1.5}$$

2. CONSTRUCTING THE ANALYTICAL SOLUTION OF THE BOUNDARY-VALUE PROBLEM

The solution of problem (1.5) has the form

$$\varphi_1(p,k,z) = \frac{iF_-(p,k,z_-)F_+(p,k,z_+)}{pW(p,k)},$$

$$F_\mp(p,k,z) = D_\lambda(\mp z\sqrt{2k/p}\,), \quad z_- = \min(z,z_0), \quad z_+ = \max(z,z_0), \tag{2.1}$$

$$p = \mu M - \omega, \qquad \lambda = (k - p - kp^2)/(2p),$$

where $D_\lambda(\tau)$ is the parabolic cylinder function satisfying the equation [10]

$$\frac{\partial^2 D_\lambda(\tau)}{\partial \tau^2} + \Big(\lambda + \frac{1}{2} - \frac{\tau^2}{4}\Big) D_\lambda(\tau) = 0, \tag{2.2}$$

$W(p,k) = -2\sqrt{\pi k}/\Gamma(-\lambda)\sqrt{p}$ is the Wronskian of the functions $F_-(p,k,z)$ and $F_+(p,k,z)$, and $\Gamma(\lambda)$ is the gamma function [11]. The function $F_-(p,k,z) \to 0$ as $z \to -\infty$ and the function $F_+(p,k,z) \to 0$ as $z \to \infty$, so the function $\varphi_1(p,k,z)$ satisfies the boundary condition of problem (1.5). The expression obtained for the Green function (2.1) is valid for $p > 0$. Then, the solution of problem (1.5) for $p < 0$ is constructed. Another couple of linearly dependent solutions of Eq. (2.2) are used for this purpose: $F_\mp(p,k,z) = D_{-\lambda-1}(\mp iz\sqrt{2k/p}\,)$ with the Wronskian $W(p,k) = -2\sqrt{\pi k}/(\Gamma(\lambda+1)\sqrt{-p}\,)$ [the value of λ is determined in expression (2.1)]. The solution obtained is denoted as $\varphi_2(p,k,z)$, then the solution of problem (1.5) for any value of p has the form

$$\varphi(p,k,z) = \begin{cases} \varphi_1(p,k,z), & p > 0, \\ \varphi_2(p,k,z), & p < 0. \end{cases} \tag{2.3}$$

It can be shown that $\varphi_2(-p,k,z) = -\varphi_1(p,k,z)$ for $p > 0$. Indeed, the function $D_{-\lambda-1}(\mp iz\sqrt{2k/p}\,)$ is an even continuation of $D_\lambda(\mp z\sqrt{2k/p}\,)$ from the semiaxis $p > 0$ to the semiaxis $p < 0$ [10]. Assuming that $p = |q|$, we have $-\lambda - 1 = (k + |q| - k|q|^2)/(2|q| - 1) = (k - |q| - k|q|^2)/(2|q|)$ for $p < 0$ and $\lambda = (k - |q| - k|q|^2)/(2|q|)$ for $p > 0$. Therefore, $D_{-\lambda-1}(\mp iz\sqrt{2k/p}\,) = D_\lambda(\mp z\sqrt{2k/p}\,)$. The function $\varphi_2(p,k,z)$ is an odd continuation of the function $\varphi_1(p,k,z)$ from the positive side of the p axis to the negative one. It can be shown that the function $\varphi(p,k,z)$ determined by Eq. (2.3) is analytical (except for poles) in the vicinity of the point $p = 0$. For this purpose, the behavior of the function $\varphi_1(p,k,z)$ as $p \to +0$ is analyzed. Then, the asymptotic representation of the Ventzel–Cramer–Brillouin of the parabolic cylinder function $D_\lambda(\sqrt{2}\,\tau)$ as $\lambda \to +\infty$ is implemented. We introduce the denotation $A(\tau) = \tau^2 - (2\lambda + 1)$, then the Ventzel–Cramer–Brillouin asymptotics of the solution of Eq. (2.2) for large values of λ and $\tau > \sqrt{2\lambda + 1}$ takes the form [11, 12]

$$D_\lambda(\sqrt{2}\,\tau) \approx \frac{\mathrm{e}^{-\sqrt{A(\tau)}}}{(A(\tau))^{1/4} B(\lambda)}, \tag{2.4}$$

where $B(\lambda)$ is determined from the condition $D_\lambda(\sqrt{2}\,\tau) \approx 2^{\lambda/2}\tau^\lambda\,\mathrm{e}^{-\tau^2/2}$ as $\tau \to \infty$. Then, $B(\lambda) = 2^{(\lambda+1)/2}\,\mathrm{e}^{(2\lambda+1)/2}$. Expression (2.4) is continued analytically into the oscillation region $|\tau| < \sqrt{2\lambda+1}$ and the use of the asymptotic representation of the gamma function as $\lambda \to +\infty$ ($\Gamma(\lambda) = \lambda^\lambda\,\mathrm{e}^{-\lambda}\,\sqrt{2\pi/\lambda}$ [12]) yields this result:

$$D_\lambda(\sqrt{2}\,\tau) \approx \frac{\sqrt{2\Gamma(\lambda+1)}}{(\pi(2\lambda+1) - \tau^2)^{1/4}} \cos\Big(\frac{1}{2}\Big(\tau\sqrt{2\lambda+1-\tau^2} + (2\lambda+1)\arcsin\Big(\frac{\tau}{\sqrt{2\lambda+1}}\Big)\Big) - \frac{\pi\lambda}{2}\Big).$$

A complement formula is used for the gamma function $\Gamma(-\lambda) = -\pi/\Gamma(\lambda+1)\sin(\pi\lambda)$ [11], and the following result is obtained for small values of p:

$$\varphi_1(p,k,z) \approx \frac{i\cos(k(z+\pi/4)/p)\cos(k(z_0-\pi/4)/p)}{k\sin(\pi k/(2p))\sqrt{1-z^2}}.$$

This expression proves the analytical nature of the function $\varphi(p,k,z)$ with respect to the variable p, and the point $p = 0$ is a nonisolated singularity, which is the extreme point of poles.

3. SOLUTION ASYMPTOTICS

Next is the integration with respect to μ in expression (1.4). The behavior of the function $\varphi(p,k,z)$ for large μ is considered, while $\lambda \approx -kp/2$. For large negative values of λ, the asymptotics of the parabolic cylinder function has the form $D_\lambda(\tau) \approx 2^{-1/2}\exp(0.5\lambda\ln(-\lambda) - 0.5\lambda - \tau\sqrt{-\lambda}\,)$ [10]. For large values of μ, the function $\varphi(p,k,z)$ can be represented in the form $\varphi(p,k,z) \approx -i\exp(-k|z-z_0|)/(2pk)$ using this asymptotics. Consequently, integral (1.4) converges with respect to μ for any fixed value of ν. Next, it is necessary to determine the poles of the function $\varphi(p,k,z)$, which are the Wronskian zeroes (gamma function zeroes) and can be calculated from the condition

$$\frac{k - p - kp^2}{2p} = n, \qquad n = 0, 1, 2, \dots\ . \tag{3.1}$$

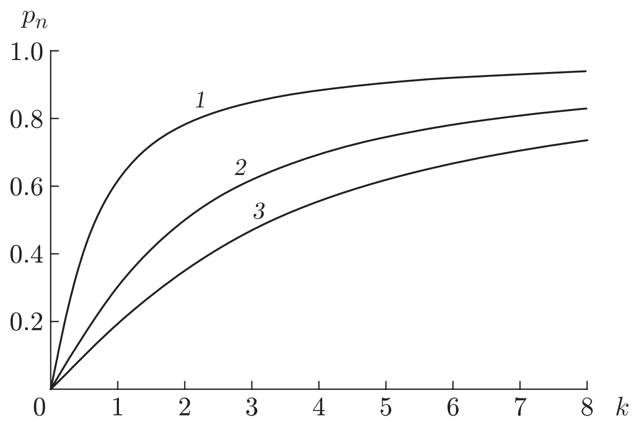

Fig. 1. Variance curves $p_n(k)$ $(n = 0, 1, 2)$: zeroth mode *1*, first mode *2*, and second mode *3*.

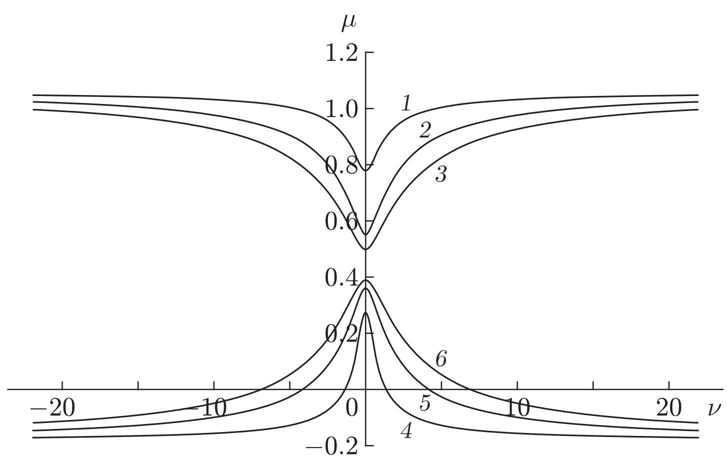

Fig. 2. Variance curves for the first three modes of μ_{n1} (*1–3*) and μ_{n2} (*4–6*): curves *1*, *2*, and *3* refer to μ_{01}, μ_{11}, and μ_{21}, respectively; curves *4*, *5*, and *6* refer to μ_{02}, μ_{12}, and μ_{22}, respectively.

Equation (3.1) has two solutions: $p_{n\pm}(k) = (-2n - 1 \pm \sqrt{(2n+1)^2 + 4k^2}\,)/(2k)$. The poles with respect to the variable μ are determined from the equation

$$|\mu M - \omega| = p_n(k) \tag{3.2}$$

$[p_n(k) \equiv p_{n+}(k)]$, which has the no real roots for $p_{n-}(k)$. In the general case, Eq. (3.2) has four real roots $\mu_{nj}(\nu)$ $(j = 1, 2, 3, 4)$. For $M > 1$, $\omega < 1$, and any value of ν, Eq. (3.2) has two series of real roots. The first series of roots $\mu_{n1}(\nu)$ for the function $\varphi_1(p, k, z)$ is a solution of the equation $\mu_{n1}(\nu)M - \omega = p_n(\sqrt{\mu_{n1}^2(\nu) + \nu^2}\,)$ and satisfies the condition $(1 + \omega)/M > \mu_{01}(\nu) > \mu_{11}(\nu) > \mu_{21}(\nu) > \ldots > \omega/M$. The second series of roots $\mu_{n2}(\nu)$ for the function $\varphi_2(p, k, z)$ is a solution of the equation $\omega - \mu_{n2}(\nu)M = p_n(\sqrt{\mu_{n2}^2(\nu) + \nu^2}\,)$ and satisfies the condition $(\omega - 1)/M < \mu_{02}(\nu) < \mu_{12}(\nu) < \mu_{22}(\nu) < \ldots < \omega/M$. Figures 1 and 2 illustrate the variance curves $p_n(k)$, $\mu_{n1}(\nu)$, and $\mu_{n2}(\nu)$ for the first three modes for $M = 1.6$ and $\omega = 0.7$ obtained in the calculations.

A perturbation method can be used to show that all poles are displaced into the lower half-plane of the complex variable μ. As the integration contour is closed for $x < 0$, it is determined that the wave field is weakly exponential: there are no real poles. We have

$$\eta(x, y, z) = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} \frac{T_{jn+} T_{jn-}}{2\sqrt{\pi}\, n! \Lambda(\mu_{nj}(\nu)) \sqrt{kR_{nj}}}\, \mathrm{e}^{-i(\mu_{nj}(\nu)x + \nu y)}\, d\nu, \tag{3.3}$$

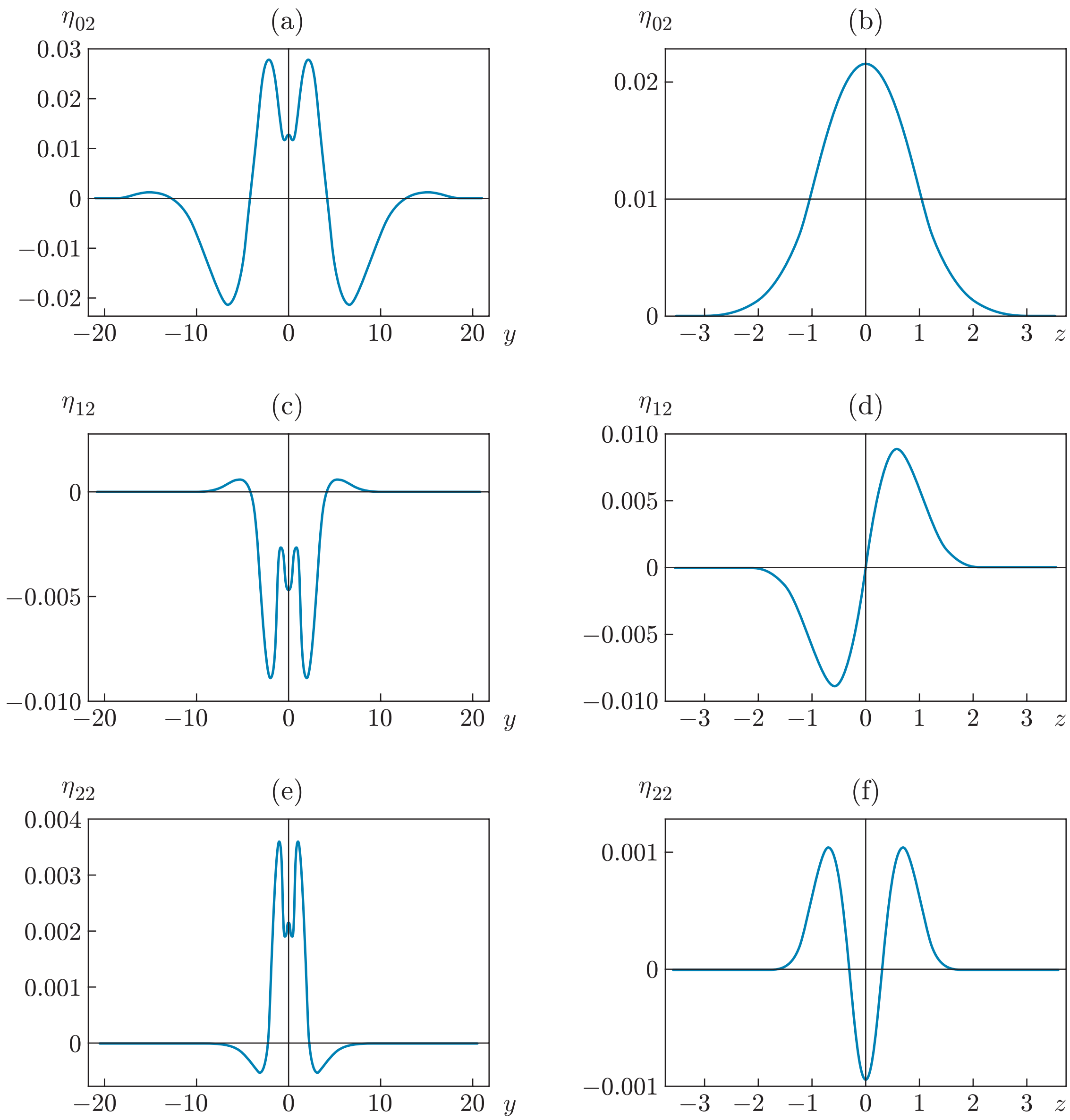

Fig. 3. First three modes η_{n2} for $x = 20$ and $z = -0.2$ (a, c, and e) and $x = 20$ and $y = 3$ (b, d, and f): (a, b) $n = 0$; (c, d) $n = 1$; (e, f) $n = 2$.

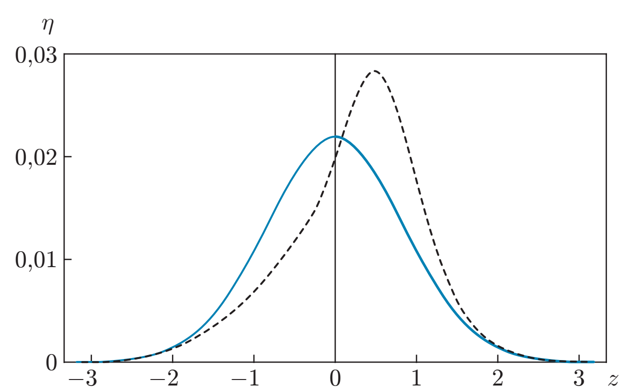

Fig. 4. First mode η_{02} (solid curve) and the sum of the modes $\eta_{02} + \eta_{12} + \eta_{22}$ (dashed curve) for $x = 20$ and $y = 3$.

$$T_{jn\pm} = D_n\Big(\pm z_\pm\sqrt{\frac{2k}{R_{nj}}}\Big), \quad R_{n1} = \mu_{n1}(\nu)M - \omega, \quad R_{n2} = \omega - \mu_{n2}(\nu)M, \quad \Lambda(\mu) = \frac{\partial\lambda}{\partial\mu}.$$

The full IGW field consists of the sum of two series of wave modes. Each series is an integral with respect to ν, which can be estimated at long distances from the perturbation source using a stationary phase method. Expression (3.3) is written as $\eta(x,y,z) = \sum_{n=0}^{\infty}(-1)^n(\eta_{n1}(x,y,z) + \eta_{n2}(x,y,z))$, and the integrals of the first series $\eta_{n1}(x,y,z)$ with variance curves $\mu_{n1}(\nu)$ are considered. The amplitude of the corresponding integrands in Eq. (3.3) is denoted as $A_{n1}(\nu,z)$. Then, the following result is obtained in the approximation of the stationary phase

$$\eta_{n1}(x,y,z) = B_{n+} + B_{n-},$$

$$B_{n\pm} = \frac{A_{n1}(\nu_\pm,z)}{\sqrt{2\pi x b_n(\nu_\pm)}}\,\mathrm{e}^{-i(\mu_{n1}(\nu_\pm)x - \nu_\pm y \pm \pi/4)}, \qquad b_n(\nu) = \frac{\partial^2\mu_{n1}(\nu)}{\partial\nu^2},$$

where $\nu_\pm$ denotes the roots of the equation $\partial\mu_{n1}(\nu)/\partial\nu = y/x$. This equation is valid in the region of the wave wedge whose opening half-angle θ is determined from the expression $\theta = \arctan(\mu_{n1}(\nu_n^*))$ [ν_n^* is the root of the equation $\partial^2\mu_{n1}(\nu)/\partial\nu^2 = 0$]. Similar estimates can be obtained for the integrals η_{n2}, and the phase pattern becomes more complex as the variance curves $\mu_{n2}(\nu)$ intersect the axis $\mu = 0$. The asymptotics describing the IGW fields far away from the perturbation sources and applicable both near the wave wedge and at a distance from it (uniform asymptotics) is expressed through the Airy function and its derivative [5, 7]. Figures 3 and 4 show the computational results for $z_0 = 0.6$ for the first three modes $\eta_{n2}(x,y,z)$. The main contribution into the full IGW field comes from the first mode, which is typical for real ocean conditions [2, 4, 6].

Figure 4 shows the first mode η_{02} and the sum of the modes.

Thus, the analytical solutions of the main boundary-value problem of internal waves, which are obtained for the model quadratic distribution of buoyancy frequencies and expressed through the parabolic cylinder functions, make it possible to construct the asymptotics of the amplitude–phase characteristics of far fields generated by internal gravity waves in a stratified medium with a variable Brunt–Väisälä frequency.

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