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Analytical Solutions of the Internal Gravity Wave Equation for a Semi-Infinite Stratified Layer of Variable Buoyancy

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Abstract—The problem of constructing asymptotics describing far-field internal gravity waves generated by an oscillating point source of perturbations moving in a vertically semi-infinite stratified layer of variable buoyancy is considered. For a model distribution of the buoyancy frequency, analytical solutions of the main boundary value problem are obtained, which are expressed in terms of Whittaker functions. An integral representation for the Green's function is obtained, and asymptotic solutions are constructed that describe the amplitude-phase characteristics of internal gravity wave fields in a semi-infinite stratified medium with a variable buoyancy frequency far away from the perturbation source.

Keywords: stratified medium, internal gravity waves, variable buoyancy frequency, Whittaker function

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INTRODUCTION

In modern scientific research, an analysis of the dynamics of internal gravity waves (IGW) in stratified natural media (ocean, the Earth's atmosphere) relies heavily on asymptotic methods for the study of analytical wave generation models [1–6]. IGW fields in such media are essentially two-dimensional and, in many cases, three-dimensional. Accordingly, an analysis of two- and three-dimensional nonstationary wave motions is a computationally complicated problem. Numerical models do not allow to efficiently compute specific physical problems in wave dynamics of the ocean and atmosphere with allowance for their actual variability, since such models are intended for rather general problems, require high computational resources, and sometimes fail to take into account the specific physical features of the problem under study, which substantially limits their practical applicability. Moreover, high-performance numerical algorithms have to be verified and validated against solutions of benchmark problems [4, 6]. In the linear approximation, available approaches to the description of excited IGW fields are based on representations of wave fields by Fourier integrals and on an asymptotic analysis of resulting solutions [5, 7]. The goal of this work is to study far fields of IGW excited by an oscillating perturbation source moving in a vertically semi-infinite stratified medium of variable buoyancy.

1. FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF SOLUTIONS

We consider the problem of far-field IGW generated by a harmonically oscillating point source of perturbations of intensity $Q = q\exp(i\omega t)$, $q = \text{const}$, that moves in a vertically semi-infinite inviscid stratified medium. The source moves at the constant speed V in the horizontal x direction, the z axis is directed upward, and the source is at a depth of $-z_0$. We consider a steady-state mode of wave oscillations. In a moving coordinate system and in the Boussinesq approximation, the equation, for example, for the vertical displacement $\eta(x, y, z)$ of isopycnals (lines of constant density with the same time-harmonic dependence) [5, 7] is given by

$$\left(i\omega + V\frac{\partial}{\partial x}\right)^2 \Delta\eta + N^2(z)\Delta_2\eta = Q\left(i\omega + V\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0},$$
$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad N^2(z) = -\frac{g}{\rho_0(z)}\frac{d\rho_0(z)}{dz}, \tag{1.1}$$

where $N^2(z)$ is the squared buoyancy frequency, $\rho_0(z)$ is the unperturbed medium density as a function of depth, g is the acceleration of gravity, and $\delta(x)$ is the Dirac delta function. In what follows, we use a model distribution of buoyancy frequency of the form

$$N^2(z) = -N_0^2(2L/z + L^2/z^2),$$

which is widely used in oceanological computations for the study of IGW dynamics in the presence of a constant thermocline (a layer with a density jump). A remarkable feature of the world ocean is the existence of a constant layer with a density jump—a region where the temperature changes rapidly and, at the same time, the buoyancy frequency is highly stable. In the model representation, the dependence of the buoyancy frequency on depth may differ from empirical formulas, which are always characterized by a maximum of $N^2(z)$ in the thermocline layer. This model buoyancy frequency distribution with a single maximum allows us to solve the problem analytically, while empirical formulas require the application of numerical methods. However, numerous studies have shown that the main qualitative results concerning the amplitude-phase characteristics of IGW in the ocean depend, as a rule, on the existence of a maximum of $N^2(z)$ in the thermocline layer rather than on a particular analytical approximation of the buoyancy frequency [1–4, 8].

The boundary conditions are specified as

$$\eta = 0 \quad \text{at} \quad z = 0, \quad \eta \to 0 \quad \text{as} \quad z \to -\infty. \tag{1.2}$$

In the dimensionless coordinates and variables $x^* = x/L$, $y^* = y/L$, $z^* = z/L$, $\eta^* = q\eta/N_0L^2$, $\omega^* = \omega/N_0$, $t^* = tN_0$, and $M = V/N_0L$, Eq. (1.1) is rewritten in the following form (hereafter, the star is omitted):

$$\left(i\omega + M\frac{\partial}{\partial x}\right)^2 \Delta\eta - (2/z + 1/z^2)\Delta_2\eta = \left(i\omega + M\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\frac{\partial\delta(z - z_0)}{\partial z_0}. \tag{1.3}$$

A solution of problem (1.2), (1.3) is sought in the form of Fourier integrals:

$$\eta(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} \varphi(\mu, \nu, z)\exp(-i(\mu x + \nu y))d\mu. \tag{1.4}$$

Then, to determine the function $\varphi(\mu, \nu, z)$, we need to solve the boundary value problem

$$\frac{\partial^2\varphi}{\partial z^2} - \frac{k^2}{p^2}(2/z + 1/z^2 + p^2)\varphi = \frac{i}{p}\frac{\partial\delta(z - z_0)}{\partial z_0},$$
$$\varphi = 0 \quad \text{at} \quad z = 0, \quad \varphi(\mu, \nu, z) \to 0 \quad \text{as} \quad z \to -\infty, \quad k^2 = \mu^2 + \nu^2, \quad p = \mu M - \omega. \tag{1.5}$$

In what follows, problem (1.5) is considered with the right-hand side $\delta(z - z_0)$ (Green's function).

2. CONSTRUCTION OF ANALYTICAL SOLUTIONS

Consider two linearly independent solutions of Eq. (1.5) with a zero right-hand side:

$$F_1(p, k, z) = W_{\alpha,\beta}(-2kz), \quad F_2(p, k, z) = M_{\alpha,\beta}(-2kz),$$

where $\alpha = k/p^2$, $\beta = (1/4 + k^2/p^2)^{1/2}$, and $W_{\alpha,\beta}$ and $M_{\alpha,\beta}$ are the Whittaker functions, which satisfy the equation

$$\frac{\partial^2 u}{\partial z^2} - (\alpha/z + (1/4 - \beta^2)/z^2 - 1/4)u = 0$$

(see [9–11]). Moreover, $F_1(p,k,z) \to 0$ as $z \to -\infty$ and $F_2(p,k,z) = 0$ at $z = 0$. Then the characteristic Green's function of problem (1.5) has the form (see [9])

$$\varphi(p,k,z) = \frac{F_1(p,k,z_-)F_2(p,k,z_+)}{Wr(p,k)}; \tag{2.1}$$

here,

$$z_- = \min(z,z_0), \quad z_+ = \max(z,z_0),$$

and $Wr(p,k)$ is the Wronskian of the functions $F_1(p,k,z_-)$ and $F_2(p,k,z_+)$ given by

$$Wr(p,k) = -\frac{2\Gamma(1+2\beta)}{\Gamma(\beta-\alpha+1/2)}, \tag{2.2}$$

where $\Gamma(x)$ is the Euler gamma function [9–11]. Consider the integration with respect to μ. The function $\varphi(p,k,z)$ given by (2.1) has simple poles at the zeros of the Wronskian, or at the poles of the gamma function in the denominator of (2.2). The point $k = 0$ is a removable singularity for $\varphi(p,k,z)$, and $\Gamma(1+2\beta) \neq 0$. The poles are determined by the condition $\beta - \alpha + 1/2 = n$, $n = 0, 1, 2, \ldots$, which yields the corresponding dispersion relation

$$p_n^2(k) = 2k\Big/\left(1 + k + 2n + \sqrt{(1+k)^2 + 4kn}\right).$$

Note that, for $p = p_n(k)$, the system of functions $M_{\alpha,\beta}(-2kz)$ forms a system of eigenfunctions that are orthogonal with weight $-(2/z + 1/z^2)$ on the interval $(-\infty, 0]$; moreover, $p_n(k)$ are the corresponding eigenvalues. Define the function

$$\lambda(\mu,\nu) = \frac{(\mu^2+\nu^2)^{1/2}}{(\mu M - \omega)^2} - \left(\frac{1}{4} + \frac{\mu^2+\nu^2}{(\mu M-\omega)^2}\right)^{1/2} - \frac{1}{2}.$$

Consider Euler's reflection formula in the form (see [10, 11])

$$1/\Gamma(-\lambda) = -\Gamma(1+\lambda)\sin\pi\lambda/\pi.$$

Then Wronskian (2.2) can be represented in the form

$$Wr(\mu,\nu) = -\frac{2k\Gamma(1+2\beta)\Gamma(1+\lambda)\sin\pi\lambda}{\pi}. \tag{2.3}$$

The zeros of expression (2.3) are determined by the relation

$$\lambda(\mu,\nu) = n, \quad n = 0, 1, 2, \ldots \tag{2.4}$$

For negative n, $\Gamma(1+\lambda)\sin\pi\lambda$ is nonzero as $\lambda \to n$. Equation (2.4) is equivalent to the equation

$$(\mu M - \omega)^2 = p_n^2(k). \tag{2.5}$$

The case of a finite-thickness stratified layer with $N^2(z) = \text{const}$ was considered in [7]. It was shown that, for $M < 1$, an equation of type (2.5) has two to four roots. In the given case, Eq. (2.5) can have four roots for $M > 1$ as well. In what follows, we assume that the values of ω and M are such that the dispersion equation (2.5) has two real roots, $\mu_n^1(\nu)$ and $\mu_n^2(\nu)$, for any ν and n, which a fortiori holds for $M > 1$ and $\omega > 1/4$.

By applying the perturbation method, it can be shown that, in the integration with respect to μ, we need to go around the poles from above. Then, for $x < 0$, closing the contour of integration from above, we see that the field is exponential small, since there are no real poles. For $x > 0$, closing the integration contour in the lower half-plane and taking into account the residues at the poles $\mu_n^1(\nu)$ and $\mu_n^2(\nu)$, we obtain

$$\eta(x,y,z) = \frac{1}{2\pi i}\sum_{j=1}^{2}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{F_1(R_n^j,k,z_-)F_2(R_n^j,k,z_+)}{2k(-1)^{-n}n!\Gamma(1+2\beta)\Lambda(\mu_n^j(\nu))}\exp(-i(\mu_n^j(\nu)x+\nu y))d\nu, \tag{2.6}$$
$$R_n^j = \mu_n^j(\nu)(\nu)M - \omega, \quad \Lambda(\mu) = \partial\lambda/\partial\mu.$$

The far total field of IGW consists of the sum of two series of wave modes, each represented in the form of an integral with respect to ν. At long distances from the moving source of perturbations, each of these integrals can be estimated using the stationary phase method. Representing expression (2.6) in the form

$$\eta(x, y, z) = \sum_{m=0}^{\infty} (J_n^1 + J_n^2),$$

we consider the integrals J_n^1 (from the first series) with dispersion curves $\mu_n^1(\nu)$. Let $A_n^1(\nu, z)$ denote the amplitude of the corresponding integrands in (2.6). Then, in the stationary phase approximation,

$$J_n^1 = B_{n+} + B_{n-},$$

$$B_{n\pm} = \frac{A_n^1(\nu_\pm, z)}{\sqrt{2\pi x b_n(\nu_\pm)}} \exp(-i(\mu_n^1(\nu_\pm)x - \nu_\pm y \pm \pi/4)), \quad b_n(\nu) = \frac{\partial^2 \mu_n^1(\nu)}{\partial \nu^2},$$

where $\nu_\pm$ are the roots of the equation

$$\frac{\partial^2 \mu_n^1(\nu)}{\partial \nu} = y/x.$$

This expression is applicable inside a wave wedge with a semi-apex angle θ determined by the relation

$$\theta = \mathrm{arctg}(\mu_n^1(\nu_n^*)),$$

where ν_n^* is a root of the equation $\frac{\partial^2 \mu_n^1(\nu)}{\partial \nu^2} = 0$. Similar estimates can also be obtained for the integrals J_n^2. A far-field IGW asymptotic representation that is applicable both near and far from the wave wedge (uniform asymptotics) is expressed in terms of the Airy function and its derivative (see [5, 7]).

CONCLUSIONS

For a model distribution of buoyancy frequency with a single thermocline maximum, analytical solutions expressed in terms of the Whittaker functions were obtained for the main boundary value problem for the internal gravity wave equation. These representations can be used to construct asymptotics of amplitude-phase characteristics of far-field IGW in a stratified medium with a variable buoyancy frequency far away from the moving oscillating source of perturbations. With the use of these asymptotics, the basic characteristics of wave fields can be efficiently computed and the resulting solutions can be qualitatively analyzed, which is important for correct formulations of mathematical models of wave dynamics in actual natural media.

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