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Analytical Solutions of the Internal Gravity Wave Equation in a Stratified Medium with Shear Flows

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Abstract—The problem of constructing internal gravity wave fields generated by an oscillating localized point source of disturbances in a stratified medium with an average shear flow is considered. A model distribution of shear flow over depth is considered, and an analytical solution of the problem is obtained in the form of a characteristic Green function expressed in terms of modified Bessel functions of imaginary index. Analytical expressions for the dispersion relations are obtained using Debye asymptotics of the modified Bessel functions. Integral representations of solutions are constructed. The wave characteristics of the excited fields are investigated depending on the basic parameters of the used stratification models, shear flows, and generation modes.

Keywords: stratified medium, internal gravity waves, buoyancy frequency, shear flows, modified Bessel functions

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INTRODUCTION

The interaction of excited waves with hydrodynamic flows occupy a special place among the wide variety of wave process of various physical natures observed in the Earth's ocean and atmosphere (see [1, 2]). The motion of the stratified medium is a major factor influencing the dynamics of internal gravity waves (IGW) in both natural conditions and engineering devices. In modern scientific research, asymptotic methods for studying analytical wave generation models are used to analyze IGW dynamics in natural stratified media with allowance for flows. In the linear approximation, the approaches used to describe wave patterns of excited IGW fields are based on representing wave fields by Fourier integrals and on their asymptotic analysis [1–5]. In actual oceanic conditions, IGW propagation has to be considered against the background of an average shear flow with a vertical velocity shear such that the velocity variations are tens of cm/s or m/s, i.e., have the same order as the maximum IGW velocities. Such flows have to produce a large effect on IGW propagation. Results of numerous studies concerning field measurements of IGW, flows, and their interaction in various World ocean regions were presented in [6–8]. IGW generation by a shear flow in the Kara Strait was considered in [9]. The flow was assumed to vary with the tidal frequency, and IGW packets were generated at the tidal frequency due to the shear instability of the flow. For the Strait of Gibraltar, similar results based on flow and IGW measurements with amplitudes of several tens of meters were obtained in [10]. In modeling IGW generation, a steeply sloped transverse ridge situated in a shear flow and a periodic tidal flow within a strait can be treated as a point source in the actual ocean [6–8]. If the horizontal scale of flow variations is much greater than the IGW lengths and the scale of time variability is much greater than the IGW periods, then a natural mathematical model is the case of horizontally uniform steady shear flows (see [1–4, 8, 11, 12]).

The goal of this work is to construct analytical solutions describing IGW fields generated by an oscillating source of disturbances in a stratified medium with allowance for shear flows.

1. FORMULATION OF THE PROBLEM

Consider a layer of a vertically stratified medium of depth H. Let $(U(z), V(z))$ be the shear flow vector at the level z. Our analysis is based on the system of fluid dynamics equations linearized with respect to the unperturbed state (see [1–4, 8, 12]):

$$\rho_0\frac{DU_1}{Dt}+\frac{\partial p}{\partial x}=0,\quad \rho_0\frac{DU_2}{Dt}+\frac{\partial p}{\partial y}=0,\quad \rho_0\frac{DW}{Dt}+\frac{\partial p}{\partial z}+\rho g=0,$$

$$\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}+\frac{\partial W}{\partial z}=0,\quad \frac{\partial \rho}{\partial t}+W\frac{\partial \rho_0}{\partial z}=0,\quad \frac{D}{Dt}=\frac{\partial}{\partial t}+U(z)\frac{\partial}{\partial x}+V(z)\frac{\partial}{\partial y},$$

where (U_1, U_2, W) are the components of the perturbed velocity, (p, ρ) are the pressure and density disturbances, and $\rho_0(z)$ is the unperturbed density of the medium. By using the Boussinesq approximation, we can obtain the following equation for the vertical velocity [1, 4, 12]:

$$\begin{gathered}\frac{D^2}{Dt^2}\Delta W-\frac{D}{Dt}\left(\frac{\partial^2 U}{\partial z^2}\frac{\partial W}{\partial x}+\frac{\partial^2 V}{\partial z^2}\frac{\partial W}{\partial y}\right)+N^2(z)\Delta_2 W=Q(t,x,y,z),\\ \Delta=\Delta_2+\frac{\partial^2}{\partial z^2},\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad N^2(z)=-\frac{g}{\rho_0(z)}\frac{d\rho_0(z)}{dz},\end{gathered}\tag{1.1}$$

where $N^2(z)$ is the squared buoyancy frequency, g is the acceleration of gravity, and $Q(t,x,y,z)$ is the distribution density of sources (if any).

The boundary conditions are specified in the form (the vertical z axis is directed upward)

$$W=0\quad\text{at}\quad z=0,-H.\tag{1.2}$$

The following assumptions are made below. Assume that the buoyancy frequency is a constant: $N(z)=N=\text{const}$; the flow is one-dimensional: $V(z)\equiv 0$; and $U(z)$ is a linear function of depth: $U(z)=U_0+(U_0-U_H)z/H$, where $U_0=U(0)$ and $U_H=U(-H)$. This hydrology model is widely used in actual oceanological computations and makes it possible to take into account the basic features of wave dynamics with allowance for actual sea density variability observed in IGW field measurements in the ocean; moreover, with the use of this hydrology, the problem can be investigated analytically [7, 8, 12]. The nonzero right-hand side of (1.1) is specified as

$$Q(t,x,y,z)=q\delta(x)\delta(y)\delta(z-z_0)\exp(i\omega t),$$

i.e., we consider the Green's function for the oscillating point source of disturbances located at the depth z_0 (see [5, 11, 13]).

Then, in the dimensionless coordinates and variables

$$x^*=\pi x/H,\quad y^*=\pi y/H,\quad z^*=\pi z/H,\quad W^*=W_{HN^2}/\pi q,\quad \omega^*=\omega/N,\quad t^*=tN,$$

$$M(z^*)=\pi U(z^*)/NH=a+bz^*,\quad a=\pi U_0/NH,\quad b=(U_0-U_H)/NH$$

we pass from (1.1), (1.2) to the following system (the star is hereafter omitted):

$$\left(\frac{\partial}{\partial x}+M(z)\frac{\partial}{\partial x}\right)^2\Delta W+\Delta_2 W=\exp(i\omega t)\delta(x)\delta(y)\delta(z-z_0),\tag{1.3}$$

$$W=0\quad\text{at}\quad z=0,-\pi.\tag{1.4}$$

The parameters are specified as $a=0.8$, $b=0.2$, $\omega=0.54$. The function $W(t,x,y,z)$ is represented in the form

$$W(t,x,y,z)=\exp(i\omega t)w(x,y,z).$$

A solution of problem (1.3), (1.4) is sought in the form of Fourier integrals:

$$w(x,y,z)=\frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}d\nu\int\limits_{-\infty}^{\infty}\varphi(\mu,\nu,z)\exp(-i(\mu x+\nu y))d\mu.\tag{1.5}$$

Then, to determine the function $\varphi(\mu,\nu,z)$, we need to solve the boundary value problem

$$\frac{\partial^2\varphi}{\partial z^2}+k^2((\omega-\mu M(z))^{-2}-1)\varphi=-\delta(z-z_0)(\omega-\mu M(z))^{-2},$$
$$\varphi(\mu,\nu,0)=\varphi(\mu,\nu,-\pi)0,\quad k^2=\mu^2+\nu^2. \tag{1.6}$$

2. CONSTRUCTION OF ANALYTICAL SOLUTIONS

As two linearly independent solutions of problem (1.6) with a zero right-hand side, we use solutions that are expressed in terms of a modified Bessel function with an imaginary index [14, 15]:

$$f_{1,2}(z)=\sqrt{2\beta(\omega-\mu M(z))}I_{\pm i\lambda}(\beta(\omega-\mu M(z)),$$

where the indices 1 and 2 correspond to plus and minus signs, respectively; $\lambda=\sqrt{\beta^2-1/4}$; and $\beta=k/b\mu$.

The functions $f_1(z)$ and $f_2(z)$ are complex conjugate. Assume that the shear flow velocity is positive over the entire depth of the stratified medium, i.e., $a>0$, $a-b\pi>0$. Additionally, the Miles stability condition for the Richardson number is assumed to hold:

$$\mathrm{Ri}=N^2\left(\frac{\partial U}{\partial z}\right)^{-2}>1/4,$$

i.e., $b^2<4$ (see [1–4, 8]). It follows that $\beta^2>1/4$ and the values of λ are real. For real λ and $|\tau|<\lambda$, the function $I_{i\lambda}(\tau)$ oscillates. For imaginary λ, the function $I_{i\lambda}(\tau)$ tends to infinity for large τ and nowhere oscillates for $\tau>0$ (see [14, 15]). For the values of λ to be real for any k,μ, it is sufficient that $b^2<4$, which coincides with the Miles condition for the Richardson number. The function

$$\varphi_1(z)=i(f_1(0)f_2(z)-f_2(0)f_1(z))$$

is real and satisfies the boundary condition at the origin. The function

$$\varphi_2(z)=i(f_1(-\pi)f_2(z)-f_2(-\pi)f_1(z))$$

is real and satisfies the boundary condition at $z=-\pi$. Then the characteristic Green function of Eq. (1.6) has the form

$$\varphi(\mu,\nu,z)=-\varphi_1(z)\varphi_2(z_0)(\omega-\mu M(z))^{-2}/V\quad\text{for}\quad z>z_0,$$
$$\varphi(\mu,\nu,z)=-\varphi_1(z_0)\varphi_2(z)(\omega-\mu M(z))^{-2}/V\quad\text{for}\quad z<z_0, \tag{2.1}$$
$$V=\varphi_1(z_0)F_2(z_0)-\varphi_2(z_0)F_1(z_0),\quad F_j(z)=\frac{\partial\varphi_j(z)}{\partial z},\quad j=1,2,$$

where the Wronskian V is independent of the variable z. Define

$$z_-=\min(z,z_0),\quad z_+=\max(z,z_0).$$

Then (2.1) can be represented in the form

$$\varphi(\mu,\nu,z)=-\varphi_1(z_+)\varphi_2(z_-)(\omega-\mu M(z))^{-2}/\varphi_1(-\pi)F_2(-\pi).$$

Now we pass to integration with respect to μ in (1.5). By applying the perturbation method, it can be shown that the contour of integration with respect to μ lies above the real axis in the complex μ plane. The amplitude of the integrand $\varphi(\mu,\nu,z)$ is an analytic function of μ outside its poles and the cut L made along the real axis of μ from C to D, where C and D are the zeros of the function $I_{\pm i\lambda}$ at $z=0$ and $z=-\pi$, respectively: $C=\omega/a$ and $D=\omega/(a-\pi b)$. In this case, $C=0.675$ and $D=3.145$. The point z_0 is at the critical level if $\omega-\mu_0M(z_0)=0$, where the corresponding point $\mu_0\in L$ if $z_0\in[0,-\pi]$. Thus, the critical values of z correspond to the points of L in the complex μ plane. The zeros of the Wronskian V are the roots of the equation $\varphi_1(-\pi)=0$. Then the dispersion relation can be represented in the form

$$I_{i\lambda}(\beta(\omega-\mu M(-\pi))I_{-i\lambda}(\beta(\omega-\mu M(0))=I_{-i\lambda}(\beta(\omega-\mu M(-\pi))I_{i\lambda}(\beta(\omega-\mu M(0)). \tag{2.2}$$

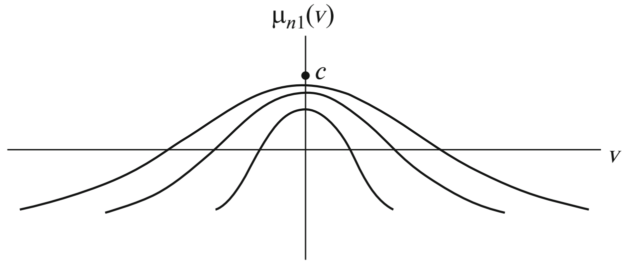

Fig. 1. Dispersion curves of the first three modes $\mu_{1n}(\nu), n = 1, 2, 3$, numbered from bottom to top.

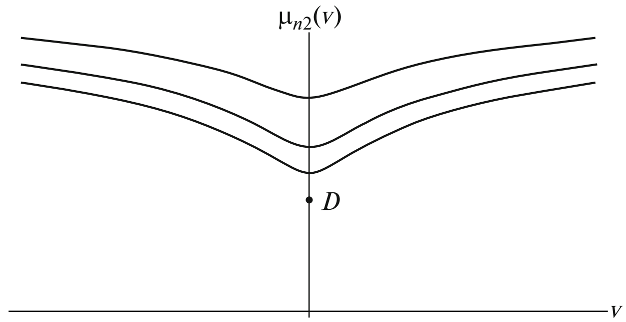

Fig. 2. Dispersion curves of the first three modes $\mu_{2n}(\nu), n = 1, 2, 3$, numbered from top to bottom.

For the given hydrology model, a dispersion relation similar to (2.2) was obtained in [12]. It was noted that the solution of this equation by analytical methods is of great difficulty. Therefore, a more complicated task is to examine the analytical properties of the resulting dispersion equation, which provides an opportunity of deriving asymptotic expressions for IGW fields in various wave generation modes. In what follows, we study the basic features of solutions to the dispersion equation (2.2) and construct asymptotic representations of its solution.

3. ANALYTICAL PROPERTIES OF DISPERSION RELATIONS

The roots of Eq. (2.2) make up two series of eigenvalues (dispersion curves) $\mu_{n1}(\nu)$ and $\mu_{n2}(\nu)$. As n grows, $\mu_{n1}(\nu)$ increase and tend to C, while $\mu_{n2}(\nu)$ decrease and tend to D. The qualitative behavior of the dispersion curves in the two series is shown in Figs. 1 and 2. Note that, for $\mu = \mu_{nj}(\nu)$ $(j = 1, 2)$, the eigenfunctions $\varphi_{n1}(\mu, \nu, z)$ and $\varphi_{n2}(\mu, \nu, z)$ of problem (1.6) are equal to each other up to a constant factor. Therefore, without loss of generality, we can assume that

$$\varphi_{n1}(\mu_{nj}(\nu), \nu, z) = \varphi_{n2}(\mu_{nj}(\nu), \nu, z) = \varphi_{nj}(\nu, z) \quad (j = 1, 2).$$

From the solution of problem (1.6), we can conclude (at $M(z) = \text{const}$) that the turning point with respect to μ, which separates the wave and nonwave zones, is determined by the relation $(\omega - \mu M)^{-2} = 1$. For a linear dependence $M(z)$, the turning point with respect to μ is determined by the relation $(\omega - \mu M(-\pi))^{-2} = 1$. Figure 3 shows the arrangement of the singular points determining the basic qualitative features of the behavior of the dispersion curves in the plane of the variables (Ψ, μ), where $\Psi = \omega - \mu M(z)$. The turning points separating the values of μ at which there are wave solutions are the points A and E in Fig. 3. The corresponding values of μ are $\mu_A = -2.67$ and $\mu_E = 8.95$. For $\mu > \mu_E$ and $\mu < \mu_A$, there are no wave disturbances.

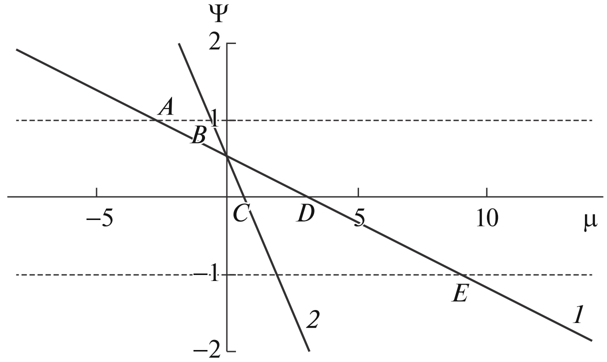

Fig. 3. Singular points of solutions of the dispersion equation for (*1*) $\Psi = \omega - \mu M(0)$ and (*2*) $\Psi = \omega - \mu M(-\pi)$.

Now the basic analytical properties of the dispersion relations following from the solution of dispersion equation (2.2) are studied in detail for various intervals of μ. For this purpose, we consider the following equation, which is satisfied by the modified Bessel functions:

$$\tau^2 Y''(\tau) + \tau Y(\tau) + (\lambda^2 - \tau^2)Y(\tau) = 0, \quad Y(\tau) = I_{\pm i\lambda}(\tau). \tag{3.1}$$

By making the substitution $Y(\tau) = U(\tau)/\sqrt{\tau}$, Eq. (3.1) can be represented in the form

$$U''(\tau) + (1/4 + \lambda^2 - \tau^2)\tau^{-2}U(\tau) = 0. \tag{3.2}$$

For $\lambda \gg 1/4$, Eq. (3.2) simplifies to

$$U''(\tau) + q(\tau)U(\tau) = 0, \quad q(\tau) = (\lambda^2 - \tau^2)\tau^{-2}. \tag{3.3}$$

Then, for oscillating solutions, the WKB asymptotic expansion of Eq. (3.3) in the case $\tau < \lambda$ has the form

$$U(\tau) \approx q(\tau)^{-1/4}\exp\left(i\int\sqrt{q(\tau)}d\tau\right),$$
$$\int\sqrt{q(\tau)}d\tau = \sqrt{\lambda^2 - \tau^2} - \frac{\lambda}{2}\ln\frac{\lambda + \sqrt{\lambda^2 - \tau^2}}{\lambda - \sqrt{\lambda^2 - \tau^2}}.$$

For exponentially growing or decaying solutions, the WKB asymptotic expansion of Eq. (3.3) for $\tau > \lambda$ has the form

$$U(\tau) \approx D_{\pm}(-q(\tau))^{-1/4}\exp\left(i\int\sqrt{-q(\tau)}d\tau\right),$$
$$\int\sqrt{-q(\tau)}d\tau = \sqrt{\tau^2 - \lambda^2} - \lambda\,\mathrm{arctg}(\sqrt{\tau^2 - \lambda^2}/\lambda).$$

In what follows, we consider only the functions $I_{-i\lambda}(\tau)$ (the function $I_{-i\lambda}(\tau)$ is the complex conjugate of $I_{i\lambda}(\tau)$ and $I_{i\lambda}(-\tau) = \exp(-\pi\lambda)I_{-i\lambda}(\tau)$). To find the multipliers $D_{\pm}$, the WKB asymptotic expansion is compared with the classical asymptotics of $I_{-i\lambda}(\tau)$ as $\tau \to \infty$:

$$\mathrm{Re}\, I_{-i\lambda}(\tau) \approx \exp(\tau)/\sqrt{2\pi\tau},$$
$$\mathrm{Im}\, I_{-i\lambda}(\tau) \approx \exp(-\tau + \pi\lambda)/2\sqrt{2\pi\tau}.$$

Then the WKB asymptotic expansion of $I_{-i\lambda}(\tau)$ for $\tau > \lambda$ has the form

$$\begin{gathered}\mathrm{Re}\, I_{-i\lambda}(\tau) \approx (\tau^2 - \lambda^2)^{-1/4}\exp(\Phi_+)/\sqrt{2\pi},\\ \mathrm{Im}\, I_{-i\lambda}(\tau) \approx (\tau^2 - \lambda^2)^{-1/4}\exp(\Phi_-)/2\sqrt{2\pi},\\ \Phi_{\pm} = \pm\sqrt{\tau^2 - \lambda^2} \mp \lambda\,\mathrm{arctg}(\sqrt{\tau^2 - \lambda^2}/\lambda) + \pi\lambda/2.\end{gathered} \tag{3.4}$$

The first formula in (3.4) is continued analytically from the domain $\tau > \lambda$ to $\tau < \lambda$ through the upper half-plane of the complex variable τ. As a result, we obtain

$$I_{-i\lambda}(\tau) \approx (\tau^2 - \lambda^2)^{-1/4} \exp(\alpha)/\sqrt{2\pi},$$
$$\alpha = -i\left(\sqrt{\lambda^2 - \tau^2} - \frac{\lambda}{2}\ln\frac{\lambda + \sqrt{\lambda^2 - \tau^2}}{\lambda - \sqrt{\lambda^2 - \tau^2}} - \pi/4\right) + \pi\lambda/2. \tag{3.5}$$

The Debye asymptotics of the modified Bessel functions $I_{-i\lambda}(\tau)$ of imaginary index (for large values of both index and argument) are obtained by replacing τ in (3.4) and (3.5) with λr (see [14, 15]):

$$\operatorname{Re} I_{-i\lambda}(\lambda r) \approx (r^2 - 1)^{-1/4} \exp(\lambda\Lambda_+)/\sqrt{2\pi\lambda},$$
$$\operatorname{Im} I_{-i\lambda}(\lambda r) \approx (r^2 - 1)^{-1/4} \exp(\lambda\Lambda_-)/2\sqrt{2\pi\lambda}, \tag{3.6}$$
$$\Lambda_\pm = \pm\sqrt{r^2 - 1} \mp \operatorname{arctg}(\sqrt{r^2 - 1}) \pm \pi/2, \quad r > 1,$$

$$I_{-i\lambda}(\lambda r) \approx (1 - r^2)^{-1/4} \exp(\Theta)/\sqrt{2\pi\lambda},$$
$$\Theta = -i\left(\lambda\sqrt{1 - r^2} - \frac{1}{2}\ln\frac{1 + \sqrt{1 - r^2}}{1 - \sqrt{1 - r^2}} - \pi/4\right) + \pi\lambda/2, \quad 0 < r < 1. \tag{3.7}$$

In the case under consideration, for large values of β ($\beta \gg 1/4$, large Richardson numbers), the functions determining the properties of the dispersion relation (2.2) have the form

$$I_{\pm i\beta}(\beta(\omega - \mu M(z)), \quad z = 0, -\pi.$$

Therefore, for large β, we can use the Debye asymptotics, since $\beta > 5 \gg 1/4$.

By virtue of the analytical properties of the modified Bessel functions, the left- and right-hand sides of the dispersion equation (2.2) are complex conjugate; therefore, to solve this dispersion equation, it suffices to set the imaginary part of the left-hand side of this equality to zero:

$$\operatorname{Im} I_{i\lambda}(\beta(\omega - \mu M(-\pi)))I_{-i\lambda}(\beta(\omega - \mu M(0))) = 0.$$

Consider a neighborhood of the point A (see Fig. 3). On the left-hand side of (2.2), the function $I_{i\lambda}(\beta(\omega - \mu M(-\pi)))$ is replaced by asymptotics (3.7), since $\omega - \mu M(-\pi) < 1$ in the neighborhood of A, while the function $I_{-i\lambda}(\beta(\omega - \mu M(0)))$ is replaced by asymptotics (3.6), since $\omega - \mu M(-\pi) < 1$ in the neighborhood of A. Since $\beta \gg 1/4$, in what follows, the values of λ in the asymptotics of modified Bessel functions are always replaced by β. Since the real part of $I_{-i\lambda}(\beta(\omega - \mu M(0)))$ grows exponentially, while its imaginary part decays exponentially with respect to μ, the contribution of this multiplier to the total phase is much less than unity and the dispersion equation (2.2) simplifies to the equation

$$\sin(\beta\Omega(r) - \pi/4) = 0, \quad \Omega(r) = \sqrt{1 - r^2} - \frac{1}{2}\ln\frac{1 + \sqrt{1 - r^2}}{1 - \sqrt{1 - r^2}}, \tag{3.7}$$

whence

$$\beta\Omega(\omega - \mu M(-\pi)) - \pi/4 = -\pi n, \quad n = 1, 2, 3, \ldots \tag{3.8}$$

Equality (3.8) contains a minus sign on the right-hand side, since $\Omega(r) < 0$. We introduce a small quantity $\varepsilon > 0$: $\varepsilon = 1 - (\omega - \mu M(-\pi))$. For small ε, the solution of (3.8) can be represented in the form

$$\frac{2\sqrt{2}}{3}\varepsilon^{2/3}\frac{\nu}{b|\mu|} = \frac{3\pi}{4} + \pi(n - 1). \tag{3.9}$$

By applying the perturbation method, the solution of Eq. (3.9) is sought in the form

$$\mu_{nl}(\nu) = \mu_0 + \sigma_{nl}\nu^{-2/3} + \gamma_{nl}\nu^{-4/3} + \ldots. \tag{3.10}$$

Substituting (3.10) into relation (3.9) and collecting like powers of ν, we obtain

$$\mu_0 = \frac{\omega - 1}{M(-\pi)}, \quad \sigma_{nl} = (3\sqrt{2}b|\mu_0|(3\pi/4 + \pi(n - 1))/4)^{2/3}/M(-\pi), \quad \gamma_{nl} = \frac{2A_{nl}^2}{3|\mu_0|}.$$

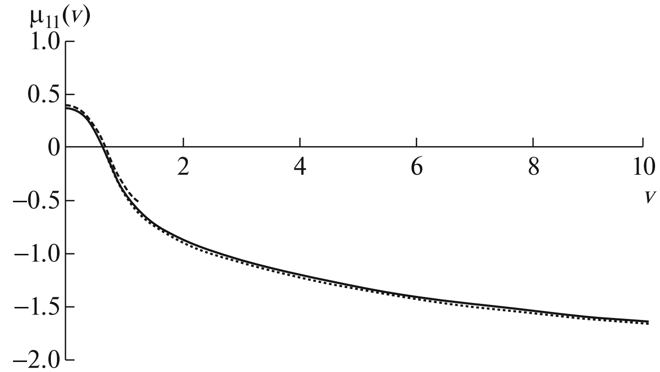

Fig. 4. Dispersion curve of the first mode $\mu_{11}(\nu)$ and its approximations.

In what follows, we everywhere set $n = 1$. The corresponding numerical values are given by $\mu_0 = -2.68$, $\sigma_{11} = 7.08$, and $\gamma_{11} = -12.46$. The constructed asymptotic expansions of the dispersion relation $\mu_{nl}(\nu)$ for large values of the parameter ν determine the features of the excited wave fields for small y.

Now we consider the interval of μ between the points A and B (ignoring the neighborhood of B); the corresponding values of μ are $\mu_A = -2.68$ and $\mu_B = -0.575$, respectively (Fig. 3). Then the dispersion relation can also be represented in the form of (3.7), but the value of $\omega - \mu M(-\pi)$ is no longer close to unity. However, Eq. (3.8) can easily be solved numerically for any n, since the left-hand side of (3.8) is a monotonic function of μ. From Eq. (3.8), we can obtain explicit analytical expressions for the dispersion relation $\nu_n(\mu)$ in the form

$$\nu_n(\mu) = |\mu|\left((\pi/4 - \pi n)^2 b^2 \Omega^{-2}(r) - 1\right)^{1/2}, \quad r = \omega - \mu M(-\pi).$$

If the interval of μ between A and B is considered with allowance for the neighborhood of B, then we need to take into account the phase addition of the multiplier $I_{-i\lambda}(\beta(\omega - \mu M(0)))$. The addition to the phase is taken into account only up to the first discontinuity of the function $\arg I_{-i\lambda}(\beta(\omega - \mu M(0)))$, i.e., until this function continuously reaches the value of π. Then Eq. (3.8) becomes

$$\beta\Omega(\omega - \mu M(-\pi)) - \pi/4 + \arg(I_{-i\lambda}(\beta(\omega - \mu M(0)))) = -\pi n, \quad n = 1, 2, 3, \ldots. \tag{3.11}$$

Figure 4 shows the dispersion curve $\mu_{11}(\nu)$ obtained by numerically solving Eq. (2.2) (solid curve) and its approximation (3.11) (dotted curve).

Consider the interval of μ from μ_B to the point C (the left boundary of the cut L in the complex plane of μ). In these values of μ, the arguments of the functions

$$I_{i\lambda}(\beta(\omega - \mu M(-\pi))), \quad I_{-i\lambda}(\beta(\omega - \mu M(0)))$$

lie in the interval $(-1,1)$, so these functions can be replaced by asymptotics (3.6). Then the dispersion equation (2.2) becomes

$$\beta(\Omega(\omega - \mu M(-\pi)) - \Omega(\omega - \mu M(0))) = \pi n, \quad n = 1, 2, 3, \ldots. \tag{3.12}$$

Figure 4 shows the dispersion curve $\mu_{11}(\nu)$ obtained by numerically solving Eq. (2.2) (solid curve) and its approximation calculated using formula (3.12) (dashed curve). From Eq. (3.12), we can also obtain an explicit analytical representation of $\nu_n(\mu)$, namely,

$$\nu_n(\mu) = |\mu|\left((\pi n b)^2 (\Omega(r_1) - \Omega(r_2))^{-2} - 1\right)^{1/2},$$
$$r_1 = \omega - \mu M(-\pi), \quad r_2 = \omega - \mu M(0).$$

The interval of μ from D (the right boundary of the cut L in the complex plane of μ) to μ_E (Fig. 3) corresponds to a family of dispersion curves $\mu_{n2}(\nu)$. In this case, the multipliers in Eq. (2.2) do not make contributions to the phase on the entire interval (D, μ_E); therefore, Eq. (3.8) has a solution for all $0 < \nu < \infty$. Figure 5 displays the dispersion curve $\mu_{21}(\nu)$ obtained by numerically solving Eq. (2.2) (solid curve) and its approximation (3.8) (dashed curve).

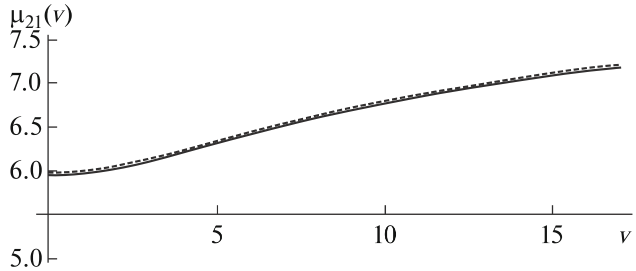

Fig. 5. Dispersion curve of the first mode $\mu_{21}(\nu)$ and its approximation.

4. ANALYTICAL REPRESENTATIONS OF WAVE FIELDS

To compute the integral with respect to μ in (1.5), we close the contour of integration in the lower half-plane and take into account the integral over the cut L and the sum of the residues at the poles $\mu = \mu_{nj}(\nu)$:

$$w(x,y,z) = \sum_{j=1}^{2}\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{0}^{\infty}A_{nj}(\nu,z,z_0)\exp(-i(\mu_{nj}(\nu)x+\nu y))d\nu + J,$$

$$A_{nj}(\nu,z,z_0) = \frac{\varphi_{nj}(\nu,z)\varphi_{nj}(\nu,z_0)}{D(\mu_{nj}(\nu),\nu,-\pi))F_{nj}(\nu,-\pi)(\omega-\mu_{nj}(\nu)M(z))^2},$$

$$J = \int_{-\infty}^{\infty}I(\nu,z,z_0)\exp(-i\nu y)d\nu,$$

$$F_{nj}(\nu,z) = \partial\varphi_{nj}(\nu,z)/\partial z, \quad D(\mu,\nu,z) = \partial\varphi_1(\mu,\nu,z)/\partial\mu, \quad j = 1,2,$$

where $I(\nu,z,z_0)$ is the integral along the bank of L. It can be shown that the contribution made by the integral along the cut bank is small as compared with the contribution of the poles $\mu = \mu_{nj}(\nu)$, so the integral J is not considered in what follows. Thus, taking into account the harmonious dependence on time, the IGW field $W(x,y,z,t)$ can be represented in the form of a sum of modes of two types:

$$W(x,y,z,t) = \sum_{n=1}^{\infty}[W_{n1}(x,y,z,t)+W_{n2}(x,y,z,t)],$$
$$W_{nj}(x,y,z,t) = \frac{1}{2\pi}\int_{0}^{\infty}A_{nj}(\nu,z,z_0)\exp(-i(\mu_{nj}(\nu)x+\nu y-\omega t))d\nu, \quad j = 1,2. \tag{4.1}$$

Far away from the source of disturbances, for large x, y, integrals (4.1) in the approximation of the stationary phase method have the form (see [5, 13])

$$W_{nj}(x,y,z,t) = Z_{nj-} + Z_{nj+},$$
$$Z_{nj\pm} = \frac{A_{nj}(\mu_{nj}(\nu_\pm),\nu_\pm,z)}{\sqrt{2\pi x(\pm s_{nj}(\nu_\pm))}}\cos(-i(\mu_{nj}(\nu_\pm)x-\nu_\pm y\pm\pi/4+\omega t)), \tag{4.2}$$
$$s_{nj}(\nu) = \frac{\partial^2\mu_{nj}(\nu)}{\partial\nu^2}, \quad j = 1,2,$$

where $\nu_\pm$ are the roots of the equation

$$\frac{\partial\mu_{nj}(\nu)}{\partial\nu} = y/x.$$

Expressions (4.2) are applicable within a corresponding wave wedge with the semi-apex angle ϑ determined by the relation $\vartheta = \arctan(\mu_{nj}(\nu_{nj}^*))$, where ν_{nj}^* is the root of the equation $s_{nj}(\nu_{nj}^*) = 0$. Approxi-

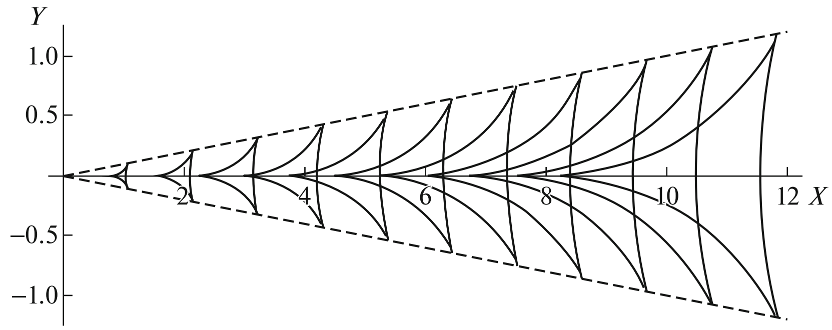

Fig. 6. Phase structure of the wave field for the dispersion curve $\mu_{11}(\nu)$.

mation (4.2) (nonuniform asymptotics) is applicable only within the wave wedges. The asymptotic expansion describing the IGW wave fields away from the source of disturbances is applicable both near and far from the wave wedges (uniform asymptotics) and is expressed in terms of the Airy function and its derivative [5, 13].

Let us describe the qualitative pattern of the excited IGW fields corresponding to two types of dispersion curves. Dispersion curves of the first type intersect the horizontal axis, the semi-apex angle of a wave wedge is less than $\pi/2$, and the wave pattern consists of a set of wedge-shaped and transverse waves. The corresponding phase structure consists of curved triangles embedded in the wave wedges with their vertex lying nearer to the origin. Since the dispersion curves of the second type are always located above the horizontal axis, they are associated with a system of wedge-shaped and longitudinal waves of simpler phase structure. The semi-apex angle for second-type waves is always less than that for first-type waves. The main contribution to the total IGW field is made by the first-type wave modes, while the amplitudes of the second-type waves are only a fraction of the first-type wave amplitudes. Figure 6 presents the computed phase structure of the excited IGW fields for the upper branch of the first-type dispersion curve $\mu_{11}(\nu)$. The dashed lines in the figure are wavefronts with the semi-apex angle ϑ, and the solid curves are lines of equal phase (determined parametrically by the parameter ν):

$$x(\nu) = \frac{\chi}{\mu_{11}(\nu) - \mu'_{11}(\nu)\nu}, \quad y(\nu) = \frac{\mu'_{11}(\nu)\chi}{\mu_{11}(\nu) - \mu'_{11}(\nu)\nu}, \quad \chi = 2\pi k, \quad k = 0,1,2,\ldots.$$

CONCLUSIONS

The problem of internal gravity wave fields generated by an oscillating localized point source of disturbances in a stratified medium with an average shear flow was considered. An analytical solution of the problem was obtained using a constant buoyancy frequency distribution and a linear dependence of the shear flow on depth. By using the chosen hydrology model, analytical expressions expressed in terms of modified Bessel functions of imaginary index were derived for the dispersion relation. Under the Miles stability condition for large Richardson numbers, analytical solutions were using the Debye asymptotics of modified Bessel functions of imaginary index. The properties of the dispersion equation were examined in detail, and the basic analytical properties of the dispersion curves were studied. Integral representations of solutions for far wave fields were constructed in the approximation of a stationary phase. The phase patterns of the excited IGW fields were numerically computed for the given model of wave generation.

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