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Far Fields of the Surface Disturbances Produced by a Pulsating Source in an Infinite-Depth Fluid

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Abstract—The problem of constructing uniform asymptotics for the far fields of surface disturbances produced by a localized pulsating source in a heavy homogeneous infinite-depth fluid is considered. The wave pattern of the excited fields is the sum of waves of two types, namely, annular and wedge-shaped. The solutions obtained describe the wave disturbances far from the pulsating source both inside and outside the Kelvin wave wedges.

Keywords: heavy fluid, pulsating source, surface disturbances, Kelvin wedge, far fields, uniform asymptotics.

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The state of the free surface of the ocean is influenced by both inhomogeneities within the water thickness (obstacles in a flow, variations in the bottom relief and the density and flow fields, etc.) and different disturbance sources [1–6]. To correctly interpret the data of the distant sounding of the sea surface the reasons resulting in some and other surface phenomena must be known. Thus, the problem of investigating the surface oscillations of a density-inhomogeneous and unsteady sea medium and bringing the simulation results into agreement with observable surface waves remains topical. To describe in detail a wide range of physical phenomena associated with the dynamics of surface disturbances of inhomogeneous and unsteady natural media fairly developed mathematical models are used [5, 6].

In certain cases the qualitative representation of a phenomenon under study can be obtained on the basis of simple asymptotic models and analytical methods of their investigation. We note the classical fluid dynamics problems of the construction of asymptotic solutions describing the evolution of surface disturbances excited by localized sources in a heavy homogeneous fluid [2, 3, 7, 8]. The model solutions derived make it possible to obtain then the asymptotic representations of the wave fields with account for the variability and unsteadiness of natural media [4–6].

The purpose of this study is to construct uniform asymptotics for the far fields of the surface disturbances excited by a localized pulsating disturbance source in a flow of heavy homogeneous infinite-depth fluid. The case of the steady disturbance source was considered in [8].

1. FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF SOLUTIONS

We will consider the steady pattern of wave disturbances on the surface of a flow of an ideal heavy infinite-depth fluid moving at a velocity V in the positive direction of the x axis. The waves are generated by a pointwise pulsating source located at a depth H (the z axis is directed upward from the undisturbed fluid). The source intensity varies in accordance with the $q = \exp(i\omega t)\exp(\varepsilon t)$ law ($-\infty < t < \infty$); in what follows, we will seek the limit of the solution, as $\varepsilon \to 0$. Due to the problem linearity, in order to calculate the field of the disturbances produced by a pulsating source of an arbitrary intensity $Q = \text{const}$ it is sufficient to multiply by Q the result obtained for the source of the unit intensity q.

The potential disturbance $\Phi(x, y, z, t)$ relative to a homogeneous flow moving at a velocity V ($\nabla\Phi = (u, v, w)$, where u, v, and w are the components of the disturbance $(V, 0, 0)$) is governed by the equation with the corresponding linearized boundary condition on the fluid surface [2, 3, 8]

$$\begin{aligned} &\Delta\Phi(x, y, z, t) = \exp(i\omega t)\exp(\varepsilon t)\delta(x)\delta(z + H), && z < 0, \\ &\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \Phi + g\frac{\partial \Phi}{\partial z} = 0, && z = 0. \end{aligned} \tag{1.1}$$

Here, Δ is the three-dimensional Laplace operator and $\delta(x)$ is the Dirac delta function. The solution of problem (1.1) is sought in the form $\Phi(x, y, z, t) = \exp(i\omega t)\exp(\varepsilon t)\varphi(x, y, z)$, where the function $\varphi(x, y, z)$ is determined from the problem

$$\begin{aligned} &\Delta\varphi(x, y, z) = \delta(x)\delta(y)\delta(z + H), && z < 0, \\ &\left(i\omega + \varepsilon + V\frac{\partial}{\partial x}\right)^2 \varphi + g\frac{\partial \varphi}{\partial z} = 0, && z = 0. \end{aligned}$$

The Fourier image of the potential $\varphi(x, y, z)$

$$\Omega(\mu, \nu, z) = \int_{-\infty}^{\infty} \exp(i\mu x)\, dx \int_{-\infty}^{\infty} \exp(i\nu y)\varphi(x, y, z)\, dy$$

is determined from the boundary value problem

$$\begin{aligned} &\frac{\partial^2 \Omega(\mu, \nu, z)}{\partial z^2} - k^2\Omega(\mu, \nu, z) = \delta(z + H), \quad z < 0, \\ &(i\omega + \varepsilon - i\mu V)^2\Omega(\mu, \nu, z) + g\frac{\partial \Omega(\mu, \nu, z)}{\partial z} = 0, \qquad z = 0, \\ &\Omega(\mu, \nu, z) \to 0, \quad z \to \infty, \qquad k^2 = \mu^2 + \nu^2, \end{aligned}$$

whose solution in the domain $-H < z < 0$ is as follows:

$$\Omega(\mu, \nu, z) = -\frac{(\omega - \mu V)^2 \sinh(kz) + gk\cosh(kz)}{k\exp(kH)((\varepsilon + i(\omega - \mu V))^2 + gk)}.$$

The free surface elevation $\eta(x, y, t)$ is related with the potential $\Phi(x, y, z, t)$ by the condition [2, 3]

$$\begin{aligned} \eta(x, y, t) &= -\frac{1}{g}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\Phi(x, y, z, t) \\ &= \frac{-\exp(i\omega t + \varepsilon t)}{g}\left(i(\omega - i\varepsilon)\varphi(x, y, z) + V\frac{\partial \varphi(x, y, z)}{\partial x}\right), \quad z = 0. \end{aligned}$$

Then the Fourier image $\Lambda(\mu, \nu, t)$ of the function $\eta(x, y, t)$ takes the form:

$$\Lambda(\mu, \nu, t) = \frac{i(\omega - \mu V)\exp(i\omega t)\exp(-kH)}{\left(\varepsilon + i(\omega - \mu V)\right)^2 + gk}.$$

In this expression the parameter ε is retained only in the denominator; this is necessary for determining the integrand pole displacement with respect to the real axis into the upper or lower half-plane. Performing the inverse Fourier transformation we obtain

$$\eta(x, y, t) = \frac{i\exp(i\omega t)}{4\pi^2}\int_{-\infty}^{\infty} \exp(-i\nu y)\, d\nu \int_{-\infty}^{\infty} \frac{(\omega - \mu V)\exp(-kH - i\mu x)\, d\mu}{(\varepsilon + i(\omega - \mu V))^2 + gk}. \tag{1.2}$$

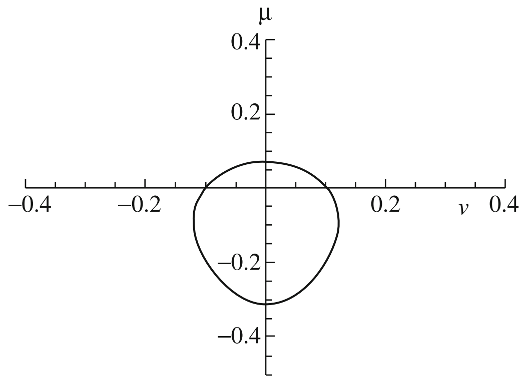

Fig. 1. Dispersion curve $k_1(\psi)$.

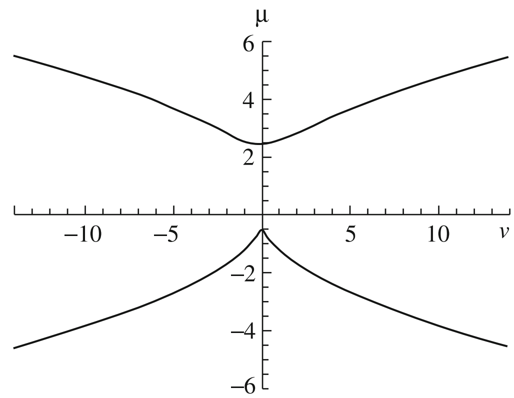

Fig. 2. Dispersion curve $k_2(\psi)$.

In the polar coordinates $(x = r\cos\alpha,\ y = r\sin\alpha)$, $(\mu = k\cos\psi,\ \nu = k\sin\psi)$ expression (1.2) takes the form:

$$\eta(r,\ \alpha,\ t) = \frac{\exp(i\omega t)}{4\pi^2 i}\int\limits_0^{2\pi} d\psi \int\limits_0^{\infty} \frac{(\omega - kV\cos\psi)k\exp(-kH - ikr\cos(\psi - \alpha))}{(\omega - i\varepsilon - kV\cos\psi)^2 - gk}\,dk. \tag{1.3}$$

In Eq. (1.3) the inner integral is calculated using the residue theorem. To do this it is necessary to determine the poles of the integrand which are the roots of the dispersion equation

$$(\omega - kV\cos\psi)^2 = gk, \tag{1.4}$$

where $k_{1,2}(\psi) = g(1 + 2m\cos\psi \mp \sqrt{1 + 4m\cos\psi})/(2V^2\cos^2\psi)$ and $m = V\omega/g$; the minus and plus signs correspond to the roots k_1 and k_2, respectively.

In what follows, we will consider the case in which the dispersion equation (1.4) has two real positive roots for any values of ψ. In order for this to happen the condition $m < 1/4$ must be fulfilled. Using the disturbance method it can be shown that at $\varepsilon > 0$ the root $k_1(\psi)$ is displaced into the lower half-plane of the complex plane k for any values of ψ. The root $k_2(\psi)$ is displaced into the lower half-plane at $\cos\psi > 0$ and into the upper half-plane at $\cos\psi < 0$. In Figs. 1 and 2 we have plotted the dispersion curves $k_1(\psi)$ and $k_2(\psi)$, respectively, in the $(\nu,\ \mu)$ plane. All numerical calculations are carried out for the following values of the parameters: $V = 2.4$ m/s, $\omega = 1\ \mathrm{s}^{-1}$, $g = 9.8\ \mathrm{m/s}^2$, and $m = 0.245$.

We will first consider the contribution made by the root $k_1(\psi)$ into expression (1.3) for $\eta(r,\ \alpha,\ t)$. The function $\eta(r,\ \alpha,\ t)$ is even with respect to the argument α; for this reason, below we will assume that $0 < \alpha < \pi$. We will rotate counterclockwise by $\pi/2$ the contour of integration with respect to the variable k under the condition that $\cos(\psi - \alpha) < 0$; in this case, the residue is not taken into account and the integral along the imaginary axis is of the order $O(1/r^2)$, as $r \to \infty$. At $\cos(\psi - \alpha) > 0$ the integration contour is rotated by $\pi/2$ clockwise and then, with account for the residue, we obtain

$$\eta_1(r,\ \alpha,\ t) = \frac{\exp(i\omega t)}{2\pi}\int\limits_{-\pi/2+\alpha}^{\pi/2+\alpha} B(k_1(\psi),\ \psi)\exp(-ik_1(\psi)r\cos(\psi - \alpha))\,d\psi,$$

$$B(k,\ \psi) = \frac{(\omega - kV\cos\psi)k\exp(-kH)}{2V\cos\psi(\omega - kV\cos\psi) + g}.$$

We will now consider the contribution made by the root $k_2(\psi)$ into expression (1.3) for $\eta(r,\ \alpha,\ t)$. In the case in which $\cos(\psi - \alpha) < 0$ the integration contour is rotated by $\pi/2$ counterclockwise. The integral along the imaginary axis is of the order $O(1/r^2)$, as $r \to \infty$, and the residue contributes only at $\cos\psi < 0$.

As a result, we obtain

$$\eta_2(r,\ \alpha,\ t) = -\frac{\exp(i\omega t)}{2\pi} \int\limits_{\pi/2+\alpha}^{3\pi/2} B(k_2(\psi),\ \psi) \exp\big(-ik_2(\psi) r \cos(\psi\ -\ \alpha)\big)\, d\psi.$$

In the case in which $\cos(\psi\ -\ \alpha) > 0$, the integration contour is rotated by $\pi/2$ clockwise. The residue makes the contribution at $\cos\psi > 0$ and the integral along the imaginary axis is also of the order $O(1/r^2)$, as $r \to \infty$. As a result, we obtain

$$\eta_3(r,\ \alpha,\ t) = \frac{\exp(i\omega t)}{2\pi} \int\limits_{-\pi/2+\alpha}^{\pi/2} B(k_2(\psi),\ \psi) \exp\big(-ik_2(\psi) r \cos(\psi\ -\ \alpha)\big)\, d\psi.$$

The free surface elevation can be represented as the sum of three terms

$$\begin{aligned}\eta(r,\ \alpha,\ t) &= \eta_1(r,\ \alpha,\ t)\ +\ \eta_2(r,\ \alpha,\ t)\ +\ \eta_3(r,\ \alpha,\ t)\\ &= \frac{\exp(i\omega t)}{2\pi}(J_1(r,\ \alpha)\ +\ J_2(r,\ \alpha)\ +\ J_3(r,\ \alpha)).\end{aligned}$$

2. DERIVATION OF THE ASYMPTOTIC SOLUTIONS

We will consider the behavior of the integrals $J_1(r,\ \alpha)$, $J_2(r,\ \alpha)$, and $J_3(r,\ \alpha)$ describing the full wave field of the surface disturbances far from the pulsating source, that is, at large values of r.

We will first evaluate the integral $J_1(r,\ \alpha)$. The phase function $q(\psi,\ \alpha) = k_1(\psi) r \cos(\psi - \alpha)$ has a single stationary point on the integration interval $(-\pi/2\ +\ \alpha,\ \pi/2\ +\ \alpha)$ for all $-\pi < \alpha < \pi$. It is determined from the equation $\partial q(\psi,\ \alpha)/\partial\psi = 0$ or $k_1(\psi)\tan\psi - k_1'(\psi)/k_1'(\psi)\tan\psi + k_1(\psi) = \tan\alpha$. We will denote this stationary point as $\psi_0(\alpha)$. It can be shown that $\partial^2 q(\psi_0(\alpha),\ \alpha)/\partial\psi^2 > 0$ for all values of α. Then the asymptotics of the integral $J_1(r,\ \alpha)$ are calculated using the stationary phase method and takes the form:

$$J_1(r,\ \alpha) = \sqrt{2\pi}\left(r\frac{\partial^2 q(\psi,\ \alpha)}{\partial\psi^2}\right)^{-1/2} B(k_1(\psi,\ \psi) \exp(i(rq(\psi,\ \alpha)\ +\ \pi/4)), \quad \psi = \psi_0(\alpha). \tag{2.1}$$

The integral $J_1(r,\ \alpha)$ is associated with the annular waves on the free surface of a fluid diverging from a pulsating oscillation source. In Fig. 3 the wave pattern of the surface disturbances is presented in the form of the function $\mathrm{Re}\, J_1(r,\ \alpha)$ calculated according to Eq. (2.1) at the source depth $H = 5$ m.

We will then consider the integral $J_3(r,\ \alpha)$. We will denote $A = \arctan\Theta$, where Θ is the maximum value with respect to ψ on the interval $(-\pi/2,\ 0)$ of the expression $k_2(\psi)\tan\psi\ -\ k_2'(\psi)/k_2'(\psi)\tan\psi + k_2(\psi)$.

The value of A determines the boundaries of the wave wedge (Kelvin wedge) described by the equation $y = \pm x\tan A$; in the case under consideration $\tan A = 0.284$. Then on the integration interval $(-\pi/2\ +\ \alpha,\ \pi/2)$ the phase function $q(\psi,\ \alpha) = k_2(\psi) r\cos(\psi\ -\ \alpha)$ has two stationary points $\psi_1(\alpha)$ and $\psi_2(\alpha)$ $(\psi_1(\alpha) < \psi_2(\alpha))$ at $\alpha < A$, none at $\alpha > A$, and the coalescing stationary points $\psi_1(\alpha) = \psi_2(\alpha)$ at $\alpha = A$. Inside the wave wedge the field can be calculated using the stationary phase method, in which case both stationary points $\psi_1(\alpha)$ and $\psi_2(\alpha)$ make the contribution; outside the wave wedge the field is exponentially small. However, as distinct from the integral $J_1(r,\ \alpha)$, the asymptotics calculated using the stationary phase method are nonuniform, since $\partial^2 q(\psi_1(A),\ A)/\partial\psi^2 = 0$. For this reason, in the vicinity of the boundary of the Kelvin wave wedge the asymptotics calculated using the stationary phase method are inapplicable.

The uniform asymptotics of the integral $J_3(r,\ \alpha)$ for large values of r and for all $0 < |\alpha| < \pi$ are constructed in the same way, as in [8, 9], and take the form:

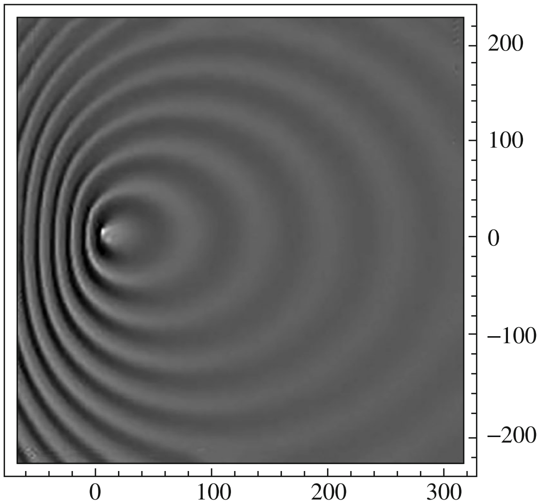

Fig. 3. Annular waves corresponding to $\mathrm{Re}J_1(r,\ \alpha)$.

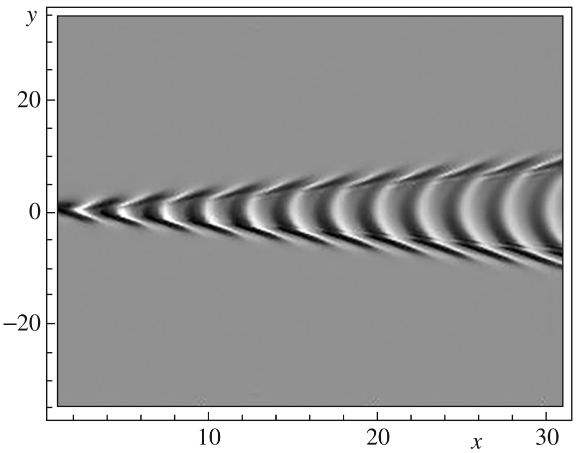

Fig. 4. Ship waves corresponding to $\mathrm{Re}J_3(r,\ \alpha)$.

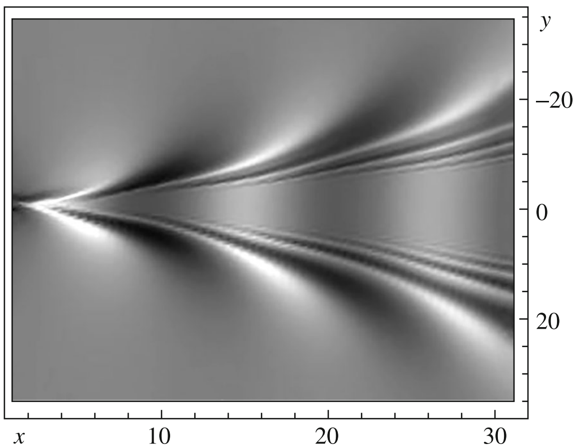

Fig. 5. Ship waves corresponding to $\mathrm{Re}J_2(r,\ \alpha)$.

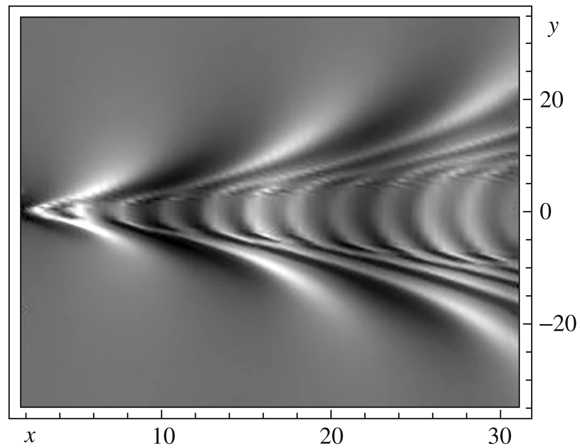

Fig. 6. Sum of two ship waves $\mathrm{Re}(J_2(r,\ \alpha) + J_3(r,\ \alpha))$.

$$
\begin{aligned}
J_3(r,\ \alpha) = {} & \frac{2\pi \exp(ir\lambda(\alpha))}{r^{1/3}}(0.5(F(\sqrt{\sigma(\alpha)}) + F(-\sqrt{\sigma(\alpha)}))\mathrm{Ai}(r^{2/3}\sigma(\alpha)) \\
& - i\frac{F(\sqrt{\sigma(\alpha)}) - F(-\sqrt{\sigma(\alpha)}}{2r^{1/3}\sqrt{\sigma(\alpha)}}\mathrm{Ai}'(r^{2/3}\sigma(\alpha))), \\
\lambda(\alpha) = {} & (q(\psi_1(\alpha),\ \alpha) + q(\psi_2(\alpha,\ \alpha))/2, \quad \sigma(\alpha) = (3(q\psi_2(\alpha),\ \alpha) - q(\psi_1(\alpha),\ \alpha))/4)^{2/3}, \\
& F(\sqrt{\sigma(\alpha)}) = f_2(\alpha)\sqrt{\frac{-2\sqrt{\sigma(\alpha)}}{T_2(\alpha)}}, \qquad F(-\sqrt{\sigma(\alpha)}) = f_2(\alpha)\sqrt{\frac{2\sqrt{\sigma(\alpha)}}{T_1(\alpha)}}, \\
& f_j(\alpha) = B\big(k_2(\psi_j(\alpha),\ \alpha)),\ \psi_j(\alpha)\big), \qquad T_j(\alpha) = \frac{\partial^2 q(\psi_j(\alpha),\ \alpha)}{\partial \psi^2}, \quad j = 1,\ 2,
\end{aligned}
\tag{2.2}
$$

where $\mathrm{Ai}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\cos(\tau t - t^3/3)\,dt$ is the Airy function and $\mathrm{Ai}'(\tau)$ is the derivative of the Airy function [9].

The nonuniform asymptotics can be obtained from Eq. (2.2), if the Airy function and its derivative are replaced by their expansions at large values of the argument. In Fig. 4 the wave pattern of the ship (wedge-shaped) waves calculated according to Eq. (2.2) is presented; these waves are described by the real part of the function $\mathrm{Re}\, J_3(r,\ \alpha)$ for the source depth $H = 0.5$ m. The integral $J_2(r,\ \alpha)$ associated with the lower part of the dispersion curve $k_2(\psi)$ in Fig. 2 is studied in the same fashion. Figure 5 presents the wave pattern described by the function $\mathrm{Re}\, J_2(r,\ \alpha)$ calculated according a formula analogous to (2.2), for the same source depth. In this case, the tangent of the wave (Kelvin) wedge semi-angle $\tan A = 0.836$. In Fig. 6 the wave pattern of the surface disturbances corresponding to the sum of the two terms $\mathrm{Re}\,(J_2(r,\ \alpha) + J_1(r,\ \alpha))$ is presented.

Summary. It is shown that the far fields of the surface disturbances produced by a localized pulsating source in a flow of a heavy infinite-depth fluid represent the system of waves of two types, namely, the annular and wedge- shaped (ship) waves. The unsteadiness of the disturbance source amplitude leads to the appearance of annular waves diverging over the fluid surface directly from the source.

In this case, the contribution into the full surface disturbance is made by two wedge-shaped (ship) waves, any of which is confined inside the corresponding Kelvin wedge. The asymptotic solutions derived make it possible to describe the far fields of the surface disturbances produced by a localized unsteady source both outside and inside the corresponding wave wedges. The asymptotics of the far fields of the wave disturbances obtained in the study allow one to effectively calculate the main characteristics of the wave fields and, moreover, to qualitatively analyze the solutions obtained, which is important for correctly formulating the mathematical models of the wave dynamics of surface disturbances in actual natural media.

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